Problem 2.19

This problem is designed to guide you through a "proof" of Plancherel's theorem, by starting with the theory of ordinary Fourier series on a *finite* interval, and allowing that interval to expand to infinity.

(a) Dirichlet's theorem says that "any" function $f(x)$ on the interval $[-a, +a]$ can be expanded as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n\pi x}{a}\right) \right].$$

Show that this can be written equivalently as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}.$$

What is c_n, in terms of a_n and b_n?

(b) Show (by appropriate modification of Fourier's trick) that

$$c_n = \frac{1}{2a} \int_{-a}^{+a} f(x) e^{-in\pi x/a}\, dx.$$

(c) Eliminate n and c_n in favor of the new variables $k = (n\pi/a)$ and $F(k) = \sqrt{2/\pi}\, a c_n$. Show that (a) and (b) now become

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k) e^{ikx} \Delta k; \quad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} f(x) e^{-ikx}\, dx,$$

where Δk is the increment in k from one n to the next.

(d) Take the limit $a \to \infty$ to obtain Plancherel's theorem. *Comment:* In view of their quite different origins, it is surprising (and delightful) that the two formulas—one for $F(k)$ in terms of $f(x)$, the other for $f(x)$ in terms of $F(k)$—have such a similar structure in the limit $a \to \infty$.

Solution

Part (a)

Suppose there's a function $f(x)$ defined on $-a < x < a$ that has a convergent Fourier series expansion.

$$f(x) = b_0 + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a} \tag{1}$$

This infinite series is the $2a$-periodic extension of $f(x)$ to the whole line.

To determine b_0, integrate both sides of equation (1) with respect to x from $-a$ to a.

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{a}\left(b_0 + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a}\right)dx \\ &= b_0 \underbrace{\int_{-a}^{a} dx}_{=2a} + a_n \sum_{n=1}^{\infty} \underbrace{\int_{-a}^{a} \sin\frac{n\pi x}{a}\,dx}_{=0} + b_n \sum_{n=1}^{\infty} \underbrace{\int_{-a}^{a} \cos\frac{n\pi x}{a}\,dx}_{=0} \\ &= b_0(2a)\end{aligned}$$

Solve for b_0.

$$b_0 = \frac{1}{2a}\int_{-a}^{a} f(x)\,dx$$

To determine b_n, multiply both sides of equation (1) by $\cos(m\pi x/a)$, where m is another integer,

$$f(x)\cos\frac{m\pi x}{a} = b_0 \cos\frac{m\pi x}{a} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a}$$

and then integrate both sides with respect to x from $-a$ to a.

$$\int_{-a}^{a} f(x)\cos\frac{m\pi x}{a}\,dx = \int_{-a}^{a}\left(b_0 \cos\frac{m\pi x}{a} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a}\right)dx$$

$$\int_{-a}^{a} f(x)\cos\frac{m\pi x}{a}\,dx = b_0 \underbrace{\int_{-a}^{a}\cos\frac{m\pi x}{a}\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-a}^{a}\sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a}\,dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-a}^{a}\cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a}\,dx}_{=a\delta_{mn}}$$

$$\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx = b_n(a)$$

Solve for b_n.

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx$$

To determine a_n, multiply both sides of equation (1) by $\sin(m\pi x/a)$, where m is another integer,

$$f(x)\sin\frac{m\pi x}{a} = b_0 \sin\frac{m\pi x}{a} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a}\sin\frac{m\pi x}{a}$$

and then integrate both sides with respect to x from $-a$ to a.

$$\int_{-a}^{a} f(x)\sin\frac{m\pi x}{a}\,dx = \int_{-a}^{a}\left(b_0 \sin\frac{m\pi x}{a} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\right)dx$$

$$\int_{-a}^{a} f(x)\sin\frac{m\pi x}{a}\,dx = b_0 \underbrace{\int_{-a}^{a}\sin\frac{m\pi x}{a}\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-a}^{a}\sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx}_{=a\delta_{mn}} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-a}^{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx}_{=0}$$

$$\int_{-a}^{a} f(x)\sin\frac{n\pi x}{a}\,dx = a_n(a)$$

Solve for a_n.

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{n\pi x}{a}\,dx$$

Euler's formula relates the complex exponential function to sine and cosine.

$$e^{iz} = \cos z + i\sin z$$

Take the complex conjugate of both sides.

$$e^{-iz} = \cos z - i\sin z$$

Solving this system of equations for sine and cosine gives

$$\begin{aligned}\cos z &= \frac{e^{iz}+e^{-iz}}{2}\\ \sin z &= \frac{e^{iz}-e^{-iz}}{2i}.\end{aligned}$$

Substitute these formulas into the Fourier series for $f(x)$.

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty}\left[a_n\sin\left(\frac{n\pi x}{a}\right) + b_n\cos\left(\frac{n\pi x}{a}\right)\right]\\
&= b_0 + \sum_{n=1}^{\infty}\left[a_n\sin\left(\frac{n\pi x}{a}\right) + b_n\cos\left(\frac{n\pi x}{a}\right)\right]\\
&= b_0 + \sum_{n=1}^{\infty}\left(a_n\frac{e^{in\pi x/a}-e^{-in\pi x/a}}{2i} + b_n\frac{e^{in\pi x/a}+e^{-in\pi x/a}}{2}\right)\\
&= b_0 + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2i}+\frac{b_n}{2}\right)e^{in\pi x/a} + \left(-\frac{a_n}{2i}+\frac{b_n}{2}\right)e^{-in\pi x/a}\right]\\
&= b_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}(b_n - ia_n)e^{in\pi x/a} + \frac{1}{2}(b_n + ia_n)e^{-in\pi x/a}\right]\\
&= b_0 + \sum_{n=1}^{\infty}\frac{1}{2}(b_n - ia_n)e^{in\pi x/a} + \sum_{n=1}^{\infty}\frac{1}{2}(b_n + ia_n)e^{-in\pi x/a}\end{aligned}$$

These coeffients are

$$\frac{1}{2}(b_n - ia_n)$$
$$\frac{1}{2}\left(\frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx - \frac{i}{a}\int_{-a}^{a} f(x)\sin\frac{n\pi x}{a}\,dx\right)$$
$$\frac{1}{2}\left[\frac{1}{a}\int_{-a}^{a} f(x)\left(\cos\frac{n\pi x}{a} - i\sin\frac{n\pi x}{a}\right)dx\right]$$
$$\frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx$$

$$\frac{1}{2}(b_n + ia_n)$$
$$\frac{1}{2}\left(\frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx + \frac{i}{a}\int_{-a}^{a} f(x)\sin\frac{n\pi x}{a}\,dx\right)$$
$$\frac{1}{2}\left[\frac{1}{a}\int_{-a}^{a} f(x)\left(\cos\frac{n\pi x}{a} + i\sin\frac{n\pi x}{a}\right)dx\right]$$
$$\frac{1}{2a}\int_{-a}^{a} f(x)e^{in\pi x/a}dx.$$

If we set $c_0 = b_0$ and

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx,$$

then

$$c_{-n} = \frac{1}{2a}\int_{-a}^{a} f(x)e^{in\pi x/a}dx,$$

and the Fourier series for $f(x)$ becomes

$$\begin{aligned}
f(x) &= b_0 + \sum_{n=1}^{\infty}\frac{1}{2}(b_n - ia_n)e^{in\pi x/a} + \sum_{n=1}^{\infty}\frac{1}{2}(b_n + ia_n)e^{-in\pi x/a} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/a} + \sum_{n=1}^{\infty} c_{-n}e^{-in\pi x/a} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}.
\end{aligned}$$

Part (b)

Start with the complex form of the Fourier series for $f(x)$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$$

To determine the coefficients c_n, multiply both sides by $e^{-im\pi x/a}$, where m is another integer,

$$f(x)e^{-im\pi x/a} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}e^{-im\pi x/a}$$

and then integrate both sides with respect to x from $-a$ to a.

$$\begin{aligned}
\int_{-a}^{a} f(x)e^{-im\pi x/a}\,dx &= \int_{-a}^{a}\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}e^{-im\pi x/a}\,dx \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{i(n-m)\pi x/a}\,dx
\end{aligned}$$

If $n \neq m$, then

$$\begin{aligned}
\int_{-a}^{a} e^{i(n-m)\pi x/a}\,dx &= \frac{a}{i(n-m)\pi}e^{i(n-m)\pi x/a}\bigg|_{-a}^{a} \\
&= \frac{a}{i(n-m)\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right] \\
&= \frac{2a}{(n-m)\pi}\left[\frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{2i}\right] \\
&= \frac{2a}{(n-m)\pi}\sin[(n-m)\pi] \\
&= 0.
\end{aligned}$$

If $n = m$, then

$$\int_{-a}^{a} e^{i(n-m)\pi x/a}\,dx = \int_{-a}^{a} dx$$
$$= 2a.$$

What this means is that every term in the infinite series is zero except for one: the $n = m$ term.

$$\int_{-a}^{a} f(x)e^{-im\pi x/a}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{i(n-m)\pi x/a}\,dx$$
$$\int_{-a}^{a} f(x)e^{-in\pi x/a}\,dx = c_n(2a)$$

Solve for c_n.

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}\,dx.$$

Part (c)

Introduce the new variable $F(n) = \sqrt{2/\pi}\,ac_n$.

$$c_n = \frac{1}{a}\sqrt{\frac{\pi}{2}}\,F(n) = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}\,dx$$

Solve for $F(n)$.

$$F(n) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-in\pi x/a}\,dx$$

Introduce the other variable $k = n\pi/a$ so that

$$\boxed{F(k) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-ikx}\,dx.}$$

Note that

$$\Delta k = k_{n+1} - k_n = \frac{(n+1)\pi}{a} - \frac{n\pi}{a} = \frac{\pi}{a}.$$

As a result, the complex Fourier series of $f(x)$ becomes

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a} \\ &= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}\,dx\right]e^{in\pi x/a} \\ &= \sum_{\frac{ak}{\pi}=-\infty}^{\infty}\left[\frac{1}{2a}\int_{-a}^{a} f(x)e^{-ikx}\,dx\right]e^{ikx} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-ikx}\,dx\right]e^{ikx}\left(\frac{\pi}{a}\right) \end{aligned}$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} F(k)e^{ikx}\Delta k.}$$

Part (d)

In the limit as $a \to \infty$, Δk becomes an infinitesimal quantity dk, and the sum turns into an integral.

$$\lim_{a \to \infty} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} \, dk$$

Additionally, the integral in the boxed formula for $F(k)$ becomes improper.

$$\lim_{a \to \infty} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$