Problem 2.2

Show that E must exceed the minimum value of $V(x)$, for every normalizable solution to the time-independent Schrödinger equation. What is the classical analog to this statement? *Hint:* Rewrite Equation 2.5 in the form

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi;$$

if $E < V_{\text{min}}$, then ψ and its second derivative always have the *same* sign—argue that such a function cannot be normalized.

Solution

The method of separation of variables involves assuming a product solution $\Psi(x,t) = \psi(x)\phi(t)$ in order to reduce the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t),$$

to two ODEs, one in x and one in t, in the special case that the potential energy is independent of t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x) &= E \end{aligned}\right\}$$

Solving the ODE in t yields $\phi(t) = Ae^{-iEt/\hbar}$, where A is an arbitrary (normalization) constant. The product solution for the Schrödinger equation then becomes $\Psi(x,t) = A\psi(x)e^{-iEt/\hbar}$. On the other hand, the ODE in x is known as the time-independent Schrödinger equation. Solve it for $\psi''(x)$.

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$$

Suppose that E is less than the minimum value of $V(x)$. Then the coefficient of ψ is positive for all x, and $d^2\psi/dx^2$ and ψ have the same sign as a result. Assuming the wave function is normalizable, the integral of $|\Psi(x,t)|^2$ over the whole line must be 1.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx \\ &= \int_{-\infty}^{\infty} \Psi(x,t)\Psi^*(x,t)\, dx \\ &= \int_{-\infty}^{\infty} [A\psi(x)e^{-iEt/\hbar}][A^*\psi(x)e^{iEt/\hbar}]\, dx \\ &= |A|^2 \int_{-\infty}^{\infty} [\psi(x)]^2\, dx \end{aligned}$$

If ψ is positive, then the concavity is upward, meaning ψ either blows up to ∞ as x increases or, if ψ becomes zero and changes concavity, skydives to $-\infty$. And if ψ is negative, then the concavity is downward, meaning ψ skydives to $-\infty$ as x increases or, if ψ becomes zero and changes concavity, blows up to ∞. In any case, this improper integral does not converge, contradicting the assumption that the wave function is normalizable. Therefore, E must exceed the minimum value of $V(x)$.