# Problem 2.24

Check the uncertainty principle for the wave function in Equation 2.132. *Hint:* Calculating $\langle p^2\rangle$ can be tricky, because the derivative of $\psi$ has a step discontinuity at $x = 0$. You may want to use the result in Problem 2.23(b). *Partial answer:* $\langle p^2\rangle = (m\alpha/\hbar)^2$.

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### Solution

Equation 2.132 has the information necessary to construct the wave function.

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}; \quad E = -\frac{m\alpha^2}{2\hbar^2} \tag{2.132}$$

The product solution is

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar} = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}e^{im\alpha^2t/(2\hbar^3)}.$$

Calculate the expectation value of $x$ at time $t$ for this bound state.

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(x)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}\left[\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}e^{-im\alpha^2t/(2\hbar^3)}\right](x)\left[\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}e^{im\alpha^2t/(2\hbar^3)}\right]dx \\
&= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty}xe^{-2m\alpha|x|/\hbar^2}\,dx \\
&= 0
\end{aligned}$$

Calculate the expectation value of $x^2$ at time $t$.

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}\left[\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}e^{-im\alpha^2t/(2\hbar^3)}\right](x^2)\left[\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}e^{im\alpha^2t/(2\hbar^3)}\right]dx \\
&= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty}x^2e^{-2m\alpha|x|/\hbar^2}\,dx \\
&= \frac{2m\alpha}{\hbar^2}\int_0^{\infty}x^2e^{-2m\alpha x/\hbar^2}\,dx \\
&= \frac{2m\alpha}{\hbar^2}\int_0^{\infty}\frac{\partial^2}{\partial\alpha^2}\left(\frac{\hbar^4}{4m^2}e^{-2m\alpha x/\hbar^2}\right)dx \\
&= \frac{\hbar^2\alpha}{2m}\frac{d^2}{d\alpha^2}\int_0^{\infty}e^{-2m\alpha x/\hbar^2}\,dx \\
&= \frac{\hbar^2\alpha}{2m}\frac{d^2}{d\alpha^2}\left(-\frac{\hbar^2}{2m\alpha}e^{-2m\alpha x/\hbar^2}\Big|_0^{\infty}\right) \\
&= \frac{\hbar^2\alpha}{2m}\frac{d^2}{d\alpha^2}\left(\frac{\hbar^2}{2m\alpha}\right) \\
&= \frac{\hbar^2\alpha}{2m}\left(\frac{\hbar^2}{m\alpha^3}\right) = \frac{\hbar^4}{2m^2\alpha^2}
\end{aligned}$$

Consequently, the standard deviation in $x$ at time $t$ is

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{\hbar^2}{\sqrt{2}m\alpha}.$$

Now calculate the expectation values of $p$ and $p^2$ at time $t$.

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\
&= -i\hbar\int_{-\infty}^{\infty}\left[\psi(x)e^{iEt/\hbar}\right]\frac{\partial}{\partial x}\left[\psi(x)e^{-iEt/\hbar}\right]dx \\
&= -i\hbar\int_{-\infty}^{\infty}\psi(x)\frac{d\psi}{dx}\,dx \\
&= -i\hbar\int_{-\infty}^{\infty}\frac{d}{dx}\left\{\frac{1}{2}[\psi(x)]^2\right\}dx \\
&= -i\hbar\left\{\frac{1}{2}[\psi(x)]^2\right\}\Big|_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

Ehrenfest's theorem confirms this.

$$\langle p\rangle = \frac{d\langle x\rangle}{dt} = 0$$

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi(x,t)\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty}\left[\psi(x)e^{iEt/\hbar}\right]\frac{\partial^2}{\partial x^2}\left[\psi(x)e^{-iEt/\hbar}\right]dx \\
&= -\hbar^2\int_{-\infty}^{\infty}\psi(x)\frac{d^2\psi}{dx^2}\,dx \\
&= -\hbar^2\Big[\underbrace{\psi(x)\frac{d\psi}{dx}\Big|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty}\frac{d\psi}{dx}\frac{d\psi}{dx}\,dx\Big] \\
&= \hbar^2\int_{-\infty}^{\infty}\left(\frac{d\psi}{dx}\right)^2 dx \\
&= \hbar^2\left[\int_{-\infty}^{0}\left(\frac{d\psi}{dx}\right)^2 dx + \int_{0}^{\infty}\left(\frac{d\psi}{dx}\right)^2 dx\right] \\
&= \hbar^2\Bigg[\int_{-\infty}^{0}\left(\frac{d}{dx}\frac{\sqrt{m\alpha}}{\hbar}e^{m\alpha x/\hbar^2}\right)^2 dx \\
&\qquad + \int_{0}^{\infty}\left(\frac{d}{dx}\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha x/\hbar^2}\right)^2 dx\Bigg] \\
&= \hbar^2\Bigg[\int_{-\infty}^{0}\left(\frac{\sqrt{m^3\alpha^3}}{\hbar^3}e^{m\alpha x/\hbar^2}\right)^2 dx \\
&\qquad + \int_{0}^{\infty}\left(-\frac{\sqrt{m^3\alpha^3}}{\hbar^3}e^{-m\alpha x/\hbar^2}\right)^2 dx\Bigg] \\
&= \frac{m^3\alpha^3}{\hbar^4}\left(\int_{-\infty}^{0}e^{2m\alpha x/\hbar^2}\,dx + \int_{0}^{\infty}e^{-2m\alpha x/\hbar^2}\,dx\right) \\
&= \frac{m^3\alpha^3}{\hbar^4}\left[\frac{\hbar^2}{2m\alpha}e^{2m\alpha x/\hbar^2}\Big|_{-\infty}^{0} + \left(-\frac{\hbar^2}{2m\alpha}\right)e^{-2m\alpha x/\hbar^2}\Big|_{0}^{\infty}\right] \\
&= \frac{m^3\alpha^3}{\hbar^4}\left(\frac{\hbar^2}{2m\alpha} + \frac{\hbar^2}{2m\alpha}\right) \\
&= \frac{m^2\alpha^2}{\hbar^2}
\end{aligned}$$

Consequently, the standard deviation in $p$ at time $t$ is

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{m\alpha}{\hbar}.$$

The uncertainty product is

$$\sigma_x\sigma_p = \left(\frac{\hbar^2}{\sqrt{2}m\alpha}\right)\left(\frac{m\alpha}{\hbar}\right) = \frac{\hbar}{\sqrt{2}},$$

which is consistent with Heisenberg's uncertainty principle ($\sigma_x\sigma_p \geq \hbar/2$).

What follows is unnecessary but instructive to go over.

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi(x)\frac{d\psi}{dx}\,dx \qquad \langle p^2\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi(x)\frac{d^2\psi}{dx^2}\,dx$$

Let's say we only got up to here, and we want to evaluate these integrals. The issue is that $\psi$ is in terms of the absolute value sign. Write it in terms of the signum (or sign) function, defined to be $-1$ if $x < 0$ and 1 if $x > 0$, which can be written in terms of the Heaviside function $\theta(x)$. It's known from Problem 2.23 that $d\theta/dx = \delta(x)$.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases} = x\begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} = x\,\operatorname{sgn} x = x[\theta(x) - \theta(-x)]$$

Taking the derivative of the absolute value sign gives

$$\begin{aligned}
\frac{d}{dx}|x| &= \frac{d}{dx}\{x[\theta(x)-\theta(-x)]\} \\
&= [\theta(x)-\theta(-x)] + x[\theta'(x) - \theta'(-x)(-1)] \\
&= [\theta(x)-\theta(-x)] + x[\delta(x) + \delta(-x)] \\
&= [\theta(x)-\theta(-x)] + x\left[\delta(x) + \frac{1}{|-1|}\delta(x)\right] \\
&= [\theta(x)-\theta(-x)] + \underbrace{2x\delta(x)}_{=\,0} \\
&= [\theta(x)-\theta(-x)] \\
&= \operatorname{sgn} x.
\end{aligned}$$

As a result,

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}$$

$$\begin{aligned}
\frac{d\psi}{dx} &= \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}\left(-\frac{m\alpha}{\hbar^2}\frac{d}{dx}|x|\right) \\
&= -\frac{\sqrt{m^3\alpha^3}}{\hbar^3}e^{-m\alpha|x|/\hbar^2}\operatorname{sgn} x
\end{aligned}$$

$$\begin{aligned}
\frac{d^2\psi}{dx^2} &= \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}\left(-\frac{m\alpha}{\hbar^2}\frac{d}{dx}|x|\right)^2 + \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}\left(-\frac{m\alpha}{\hbar^2}\frac{d^2}{dx^2}|x|\right) \\
&= \frac{\sqrt{m^5\alpha^5}}{\hbar^5}e^{-m\alpha|x|/\hbar^2}(\operatorname{sgn} x)^2 - \frac{\sqrt{m^3\alpha^3}}{\hbar^3}e^{-m\alpha|x|/\hbar^2}\frac{d}{dx}[\theta(x)-\theta(-x)] \\
&= \frac{\sqrt{m^5\alpha^5}}{\hbar^5}e^{-m\alpha|x|/\hbar^2} - \frac{\sqrt{m^3\alpha^3}}{\hbar^3}e^{-m\alpha|x|/\hbar^2}[2\delta(x)].
\end{aligned}$$

Now evaluate $\langle p \rangle$.

$$\begin{aligned}
\langle p \rangle &= -i\hbar \int_{-\infty}^{\infty} \psi(x) \frac{d\psi}{dx}\, dx \\
&= -i\hbar \int_{-\infty}^{\infty} \left( \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \right) \left( -\frac{\sqrt{m^3\alpha^3}}{\hbar^3} e^{-m\alpha|x|/\hbar^2} \operatorname{sgn} x \right) dx \\
&= i\frac{m^2\alpha^2}{\hbar^3} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2} \operatorname{sgn} x\, dx \\
&= i\frac{m^2\alpha^2}{\hbar^3} \left[ \int_{-\infty}^{0} e^{-2m\alpha(-x)/\hbar^2} (-1)\, dx + \int_{0}^{\infty} e^{-2m\alpha(x)/\hbar^2} (1)\, dx \right] \\
&= i\frac{m^2\alpha^2}{\hbar^3} \left( -\int_{-\infty}^{0} e^{2m\alpha x/\hbar^2}\, dx + \int_{0}^{\infty} e^{-2m\alpha x/\hbar^2}\, dx \right) \\
&= i\frac{m^2\alpha^2}{\hbar^3} \left[ -\left( \frac{\hbar^2}{2m\alpha} \right) e^{2m\alpha x/\hbar^2} \Big|_{-\infty}^{0} + \left( -\frac{\hbar^2}{2m\alpha} \right) e^{-2m\alpha x/\hbar^2} \Big|_{0}^{\infty} \right] \\
&= i\frac{m^2\alpha^2}{\hbar^3} \left( -\frac{\hbar^2}{2m\alpha} + \frac{\hbar^2}{2m\alpha} \right) \\
&= 0
\end{aligned}$$

Finally, evaluate $\langle p^2 \rangle$.

$$\begin{aligned}
\langle p^2 \rangle &= -\hbar^2 \int_{-\infty}^{\infty} \psi(x) \frac{d^2\psi}{dx^2}\, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \left( \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \right) \left[ \frac{\sqrt{m^5\alpha^5}}{\hbar^5} e^{-m\alpha|x|/\hbar^2} - \frac{2\sqrt{m^3\alpha^3}}{\hbar^3} e^{-m\alpha|x|/\hbar^2} \delta(x) \right] dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \left[ \frac{m^3\alpha^3}{\hbar^6} e^{-2m\alpha|x|/\hbar^2} - \frac{2m^2\alpha^2}{\hbar^4} e^{-2m\alpha|x|/\hbar^2} \delta(x) \right] dx \\
&= -\hbar^2 \left[ \frac{m^3\alpha^3}{\hbar^6} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2}\, dx - \frac{2m^2\alpha^2}{\hbar^4} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2} \delta(x)\, dx \right] \\
&= -\hbar^2 \left[ \frac{m^3\alpha^3}{\hbar^6} \left( \int_{-\infty}^{0} e^{2m\alpha x/\hbar^2}\, dx + \int_{0}^{\infty} e^{-2m\alpha x/\hbar^2}\, dx \right) - \frac{2m^2\alpha^2}{\hbar^4} e^0 \right] \\
&= -\hbar^2 \left\{ \frac{m^3\alpha^3}{\hbar^6} \left[ \frac{\hbar^2}{2m\alpha} e^{2m\alpha x/\hbar^2} \Big|_{-\infty}^{0} + \left( -\frac{\hbar^2}{2m\alpha} \right) e^{-2m\alpha x/\hbar^2} \Big|_{0}^{\infty} \right] - \frac{2m^2\alpha^2}{\hbar^4} \right\} \\
&= -\hbar^2 \left[ \frac{m^3\alpha^3}{\hbar^6} \left( \frac{\hbar^2}{2m\alpha} + \frac{\hbar^2}{2m\alpha} \right) - \frac{2m^2\alpha^2}{\hbar^4} \right] \\
&= -\frac{m^3\alpha^3}{\hbar^4} \left( \frac{\hbar^2}{m\alpha} \right) + \frac{2m^2\alpha^2}{\hbar^2} \\
&= -\frac{m^2\alpha^2}{\hbar^2} + \frac{2m^2\alpha^2}{\hbar^2} \\
&= \frac{m^2\alpha^2}{\hbar^2}
\end{aligned}$$