Problem 2.27

Consider the *double* delta-function potential

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)],$$

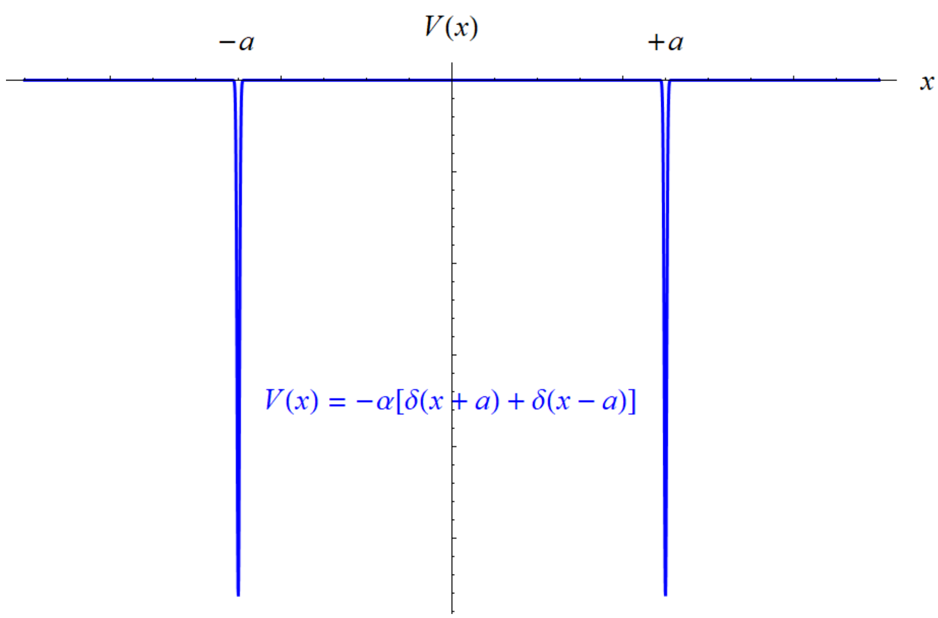
where α and a are positive constants.

(a) Sketch this potential.

(b) How many bound states does it possess? Find the allowed energies, for $\alpha = \hbar^2/ma$ and for $\alpha = \hbar^2/4ma$, and sketch the wave functions.

(c) What are the bound state energies in the limiting cases (i) $a \to 0$ and (ii) $a \to \infty$ (holding α fixed)? Explain why your answers are reasonable, by comparison with the single delta-function well.

Solution



The governing equation for the wave function $\Psi(x,t)$ is the Schrödinger equation.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

If $V(x,t) = -\alpha[\delta(x+a) + \delta(x-a)]$, then it reduces to

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \alpha[\delta(x+a) + \delta(x-a)]\Psi(x,t),$$

which can be solved by the method of separation of variables because the PDE and its associated boundary conditions (Ψ and its derivatives go to zero as $x \to \pm\infty$) are linear and homogeneous. Since the eigenstates and their energies are of interest, this method is opted for. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE.

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] - \alpha[\delta(x+a) + \delta(x-a)][\psi(x)\phi(t)]$$

Evaluate the derivatives.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) - \alpha[\delta(x+a)+\delta(x-a)]\psi(x)\phi(t)$$

Divide both sides by $\psi(x)\phi(t)$ in order to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha[\delta(x+a)+\delta(x-a)]$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha[\delta(x+a)+\delta(x-a)] = E$$

As a result of using the method of separation of variables, the Schrödinger equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha[\delta(x+a)+\delta(x-a)] &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions are satisfied are called the eigenvalues (or eigenenergies in this context), and the nontrivial solutions associated with them are called the eigenfunctions (or eigenstates in this context). The ODE in x is known as the time-independent Schrödinger equation (TISE) and can be written as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left\{\alpha[\delta(x+a)+\delta(x-a)]+E\right\}\psi(x) = 0. \tag{1}$$

If $x \neq -a, a$, then it simplifies to

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0, \quad x \neq -a, a.$$

Bound states have energy $E < 0$; in this case, the general solution is

$$\psi(x) = \begin{cases} C_1 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) + C_2 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x < -a \\ C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) + C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } -a < x < a \\ C_5 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) + C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x > a \end{cases}.$$

In order to prevent $\Psi(x,t) = \psi(x)\phi(t)$ from blowing up as $x \to \pm\infty$, set $C_2 = 0$ and $C_5 = 0$.

$$\psi(x) = \begin{cases} C_1 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x < -a \\ C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) + C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } -a < x < a \\ C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x > a \end{cases}.$$

Two more of the constants can be determined by requiring the wave function [and consequently $\psi(x)$] to be continuous across $x = -a$ and $x = a$.

$$\lim_{x\to -a^-}\psi(x) = \lim_{x\to -a^+}\psi(x): \quad C_1 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) = C_3 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) + C_4 \exp\left(\frac{\sqrt{-2mE}}{\hbar}a\right)$$

$$\lim_{x\to +a^-}\psi(x) = \lim_{x\to +a^+}\psi(x): \quad C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) = C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar}a\right) + C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right)$$

Another condition can be obtained by integrating both sides of equation (1) with respect to x from $-a-\epsilon$ to $-a+\epsilon$, where ϵ is a really small positive number.

$$\int_{-a-\epsilon}^{-a+\epsilon} \left\{ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \{\alpha[\delta(x+a)+\delta(x-a)]+E\}\,\psi(x) \right\} dx = \int_{-a-\epsilon}^{-a+\epsilon} 0\,dx$$

$$\int_{-a-\epsilon}^{-a+\epsilon} \frac{d^2\psi}{dx^2}\,dx + \frac{2m}{\hbar^2}\left\{ \alpha \int_{-a-\epsilon}^{-a+\epsilon} [\delta(x+a)+\delta(x-a)]\psi(x)\,dx + E\int_{-a-\epsilon}^{-a+\epsilon} \psi(x)\,dx \right\} = 0$$

$$\left.\frac{d\psi}{dx}\right|_{-a-\epsilon}^{-a+\epsilon} + \frac{2m}{\hbar^2}\left[\alpha\psi(-a) + E\psi(-a)\int_{-a-\epsilon}^{-a+\epsilon} dx \right] = 0$$

$$\left.\frac{d\psi}{dx}\right|_{-a-\epsilon}^{-a+\epsilon} + \frac{2m}{\hbar^2}\left[\alpha\psi(-a) + E\psi(-a)(2\epsilon) \right] = 0$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{-a^-}^{-a^+} + \frac{2m\alpha}{\hbar^2}\psi(-a) = 0$$

$$\frac{2m\alpha}{\hbar^2}\psi(-a) = \lim_{x\to -a^-}\frac{d\psi}{dx} - \lim_{x\to -a^+}\frac{d\psi}{dx}$$

$$\frac{2m\alpha}{\hbar^2} C_1 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) = \frac{\sqrt{-2mE}}{\hbar} C_1 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) - \left[\frac{\sqrt{-2mE}}{\hbar} C_3 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) - \frac{\sqrt{-2mE}}{\hbar} C_4 \exp\left(\frac{\sqrt{-2mE}}{\hbar}a\right)\right]$$

The final condition is obtained by integrating both sides of equation (1) with respect to x from $a-\epsilon$ to $a+\epsilon$.

$$\int_{a-\epsilon}^{a+\epsilon} \left\{ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \{\alpha[\delta(x+a)+\delta(x-a)]+E\}\,\psi(x) \right\} dx = \int_{a-\epsilon}^{a+\epsilon} 0\,dx$$

$$\int_{a-\epsilon}^{a+\epsilon} \frac{d^2\psi}{dx^2}\,dx + \frac{2m}{\hbar^2}\left\{ \alpha \int_{a-\epsilon}^{a+\epsilon} [\delta(x+a)+\delta(x-a)]\psi(x)\,dx + E\int_{a-\epsilon}^{a+\epsilon} \psi(x)\,dx \right\} = 0$$

$$\left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon} + \frac{2m}{\hbar^2}\left[\alpha\psi(a) + E\psi(a)\int_{a-\epsilon}^{a+\epsilon} dx \right] = 0$$

$$\left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon} + \frac{2m}{\hbar^2}\left[\alpha\psi(a) + E\psi(a)(2\epsilon) \right] = 0$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{a^-}^{a^+} + \frac{2m\alpha}{\hbar^2}\psi(a) = 0$$

$$\frac{2m\alpha}{\hbar^2}\psi(a) = \lim_{x\to a^-}\frac{d\psi}{dx} - \lim_{x\to a^+}\frac{d\psi}{dx}$$

$$\frac{2m\alpha}{\hbar^2} C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right) = \left[\frac{\sqrt{-2mE}}{\hbar} C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar}a\right) - \frac{\sqrt{-2mE}}{\hbar} C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right)\right] + \frac{\sqrt{-2mE}}{\hbar} C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}a\right)$$

Introduce the constant,

$$\kappa = \frac{\sqrt{-2mE}}{\hbar},$$

so that the four equations involving C_1, C_3, C_4, and C_6 are more manageable.

$$\begin{cases} C_1 e^{-\kappa a} = C_3 e^{-\kappa a} + C_4 e^{\kappa a} \\ C_6 e^{-\kappa a} = C_3 e^{\kappa a} + C_4 e^{-\kappa a} \\ \frac{2m\alpha}{\hbar^2} C_1 e^{-\kappa a} = \kappa C_1 e^{-\kappa a} - \left(\kappa C_3 e^{-\kappa a} - \kappa C_4 e^{\kappa a}\right) \\ \frac{2m\alpha}{\hbar^2} C_6 e^{-\kappa a} = \left(\kappa C_3 e^{\kappa a} - \kappa C_4 e^{-\kappa a}\right) + \kappa C_6 e^{-\kappa a} \end{cases}$$

Focus on these last two equations.

$$\begin{cases} \left(\frac{2m\alpha}{\hbar^2} - \kappa\right) C_1 e^{-\kappa a} = -\kappa C_3 e^{-\kappa a} + \kappa C_4 e^{\kappa a} \\ \left(\frac{2m\alpha}{\hbar^2} - \kappa\right) C_6 e^{-\kappa a} = \kappa C_3 e^{\kappa a} - \kappa C_4 e^{-\kappa a} \end{cases}$$

$$\begin{cases} \left(\frac{2m\alpha}{\hbar^2} - \kappa\right) \left(C_3 e^{-\kappa a} + C_4 e^{\kappa a}\right) = -\kappa C_3 e^{-\kappa a} + \kappa C_4 e^{\kappa a} \\ \left(\frac{2m\alpha}{\hbar^2} - \kappa\right) \left(C_3 e^{\kappa a} + C_4 e^{-\kappa a}\right) = \kappa C_3 e^{\kappa a} - \kappa C_4 e^{-\kappa a} \end{cases}$$

$$\begin{cases} \frac{2m\alpha}{\hbar^2} C_3 e^{-\kappa a} + \frac{2m\alpha}{\hbar^2} C_4 e^{\kappa a} - \cancel{\kappa C_3 e^{-\kappa a}} - \kappa C_4 e^{\kappa a} = -\cancel{\kappa C_3 e^{-\kappa a}} + \kappa C_4 e^{\kappa a} \\ \frac{2m\alpha}{\hbar^2} C_3 e^{\kappa a} + \frac{2m\alpha}{\hbar^2} C_4 e^{-\kappa a} - \kappa C_3 e^{\kappa a} - \cancel{\kappa C_4 e^{-\kappa a}} = \kappa C_3 e^{\kappa a} - \cancel{\kappa C_4 e^{-\kappa a}} \end{cases}$$

$$\begin{cases} \frac{2m\alpha}{\hbar^2} C_3 e^{-\kappa a} + \frac{2m\alpha}{\hbar^2} C_4 e^{\kappa a} = 2\kappa C_4 e^{\kappa a} \\ \frac{2m\alpha}{\hbar^2} C_3 e^{\kappa a} + \frac{2m\alpha}{\hbar^2} C_4 e^{-\kappa a} = 2\kappa C_3 e^{\kappa a} \end{cases}$$

$$\begin{cases} C_3 e^{-2\kappa a} + C_4 = \frac{\hbar^2 \kappa}{m\alpha} C_4 \\ C_3 + C_4 e^{-2\kappa a} = \frac{\hbar^2 \kappa}{m\alpha} C_3 \end{cases}$$

$$\begin{cases} C_3 = \left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} \\ C_4 = \left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right) C_3 e^{2\kappa a} \end{cases}$$

Substitute this first equation into the second one.

$$\begin{aligned} C_4 &= \left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right) \left[\left(\frac{\hbar^2 k}{m\alpha} - 1\right) C_4 e^{2\kappa a}\right] e^{2\kappa a} \\ &= \left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right)^2 C_4 e^{4\kappa a} \end{aligned}$$

Assume that $C_4 \neq 0$ and divide both sides by C_4.

$$1 = \left(\frac{\hbar^2\kappa}{m\alpha} - 1\right)^2 e^{4\kappa a} \tag{2}$$

If $\alpha = \hbar^2/(ma)$, then

$$1 = (\kappa a - 1)^2 e^{4\kappa a} \quad \Rightarrow \quad \begin{cases} \kappa a \approx 0.796812 & \rightarrow & \frac{\sqrt{-2mE}}{\hbar}a \approx 0.796812 & \rightarrow & E \approx -\frac{0.317455\hbar^2}{ma^2} \\ \kappa a = 0 & \rightarrow & \frac{\sqrt{-2mE}}{\hbar}a = 0 & \rightarrow & E = 0 \\ \kappa a \approx 1.10886 & \rightarrow & \frac{\sqrt{-2mE}}{\hbar}a \approx 1.10886 & \rightarrow & E \approx -\frac{0.614783\hbar^2}{ma^2} \end{cases}.$$

Since there are two negative energies, there are two bound states for this value of α. In particular, for $E \approx -0.317455\hbar^2/(ma^2)$, the corresponding eigenstate is

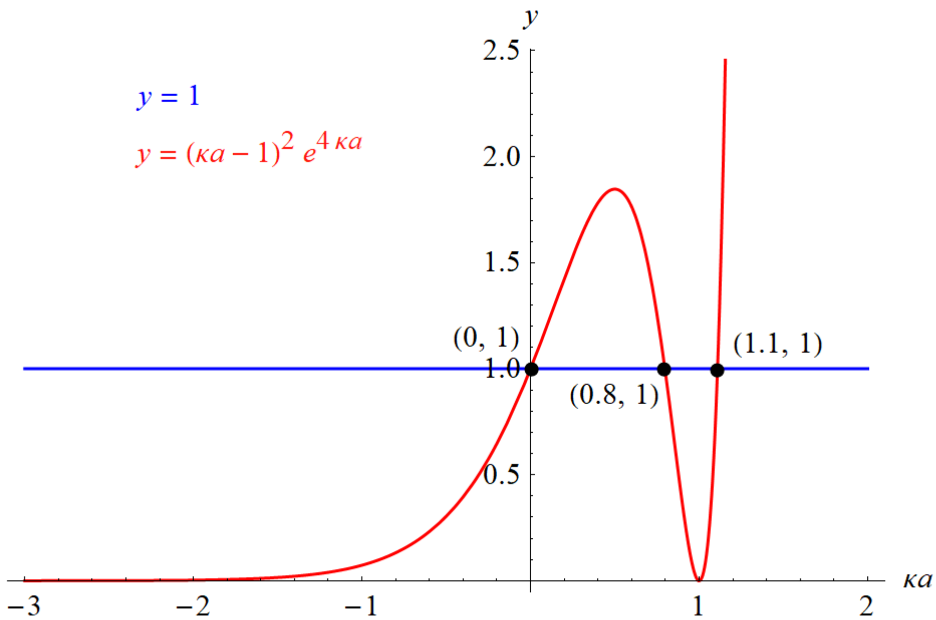
$$\psi(x) = \begin{cases} C_1 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x \leq -a \\ C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) + C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } -a \leq x \leq a \\ C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} (C_3 + C_4 e^{2\kappa a}) \exp\left(\frac{0.796812}{a}x\right) & \text{if } x \leq -a \\ C_3 \exp\left(\frac{0.796812}{a}x\right) + C_4 \exp\left(-\frac{0.796812}{a}x\right) & \text{if } -a \leq x \leq a \\ (C_3 e^{2\kappa a} + C_4) \exp\left(-\frac{0.796812}{a}x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} \left[\left(\frac{\hbar^2\kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} + C_4 e^{2\kappa a}\right] \exp\left(\frac{0.796812}{a}x\right) & \text{if } x \leq -a \\ \left(\frac{\hbar^2\kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} \exp\left(\frac{0.796812}{a}x\right) + C_4 \exp\left(-\frac{0.796812}{a}x\right) & \text{if } -a \leq x \leq a \\ \left[\left(\frac{\hbar^2\kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} e^{2\kappa a} + C_4\right] \exp\left(-\frac{0.796812}{a}x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} C_4 \left(\kappa a e^{2\kappa a}\right) \exp\left(\frac{0.796812}{a}x\right) & \text{if } x \leq -a \\ C_4 \left[(\kappa a - 1)e^{2\kappa a} \exp\left(\frac{0.796812}{a}x\right) + \exp\left(-\frac{0.796812}{a}x\right)\right] & \text{if } -a \leq x \leq a \\ C_4 \left[(\kappa a - 1)e^{4\kappa a} + 1\right] \exp\left(-\frac{0.796812}{a}x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} 3.92155 C_4 \exp\left(\frac{0.796812}{a}x\right) & \text{if } x \leq -a \\ C_4 \left[-\exp\left(\frac{0.796812}{a}x\right) + \exp\left(-\frac{0.796812}{a}x\right)\right] & \text{if } -a \leq x \leq a \\ -3.92155 C_4 \exp\left(-\frac{0.796812}{a}x\right) & \text{if } x \geq a \end{cases}.$$



The constant $C_4 \neq 0$ is arbitrary mathematically because the TISE is homogeneous, but in order to make sense of $\psi(x)$ physically, C_4 must be chosen so that the integral of $[\psi(x)]^2$ over the whole line is 1. In other words, C_4 is a normalization constant.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} [\psi(x)]^2\, dx \\
&= \int_{-\infty}^{-a} [\psi(x)]^2\, dx + \int_{-a}^{a} [\psi(x)]^2\, dx + \int_{a}^{\infty} [\psi(x)]^2\, dx \\
&= \int_{-\infty}^{-a} \left[3.92155 C_4 \exp\left(\frac{0.796812}{a}x\right)\right]^2 dx + \int_{-a}^{a} C_4^2 \left[-\exp\left(\frac{0.796812}{a}x\right) + \exp\left(-\frac{0.796812}{a}x\right)\right]^2 dx \\
&\qquad + \int_{a}^{\infty} \left[-3.92155 C_4 \exp\left(-\frac{0.796812}{a}x\right)\right]^2 dx \\
&= 5.84311 a C_4^2
\end{aligned}$$

Solve for C_4.

$$C_4 \approx \frac{0.413693}{\sqrt{a}}$$

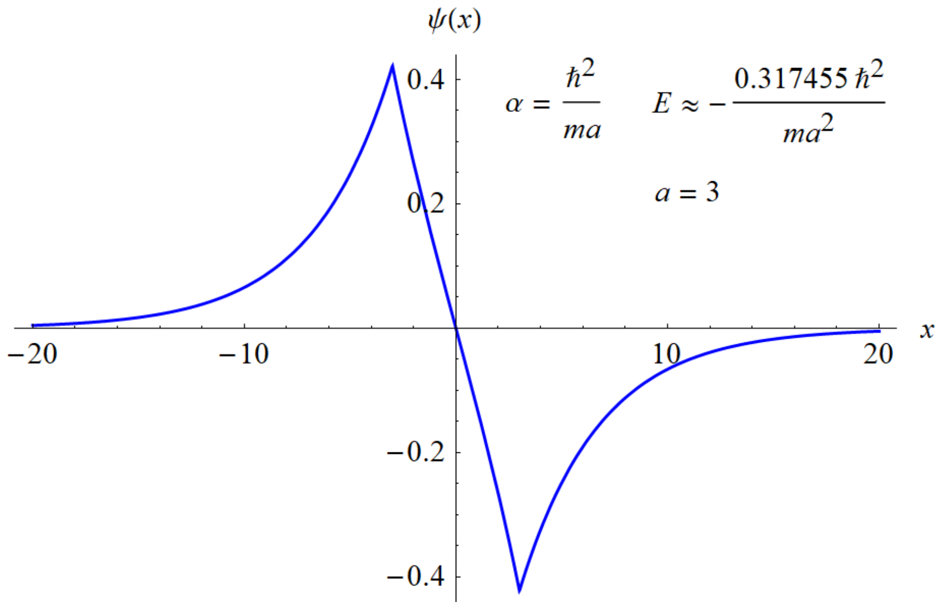
Therefore, the bound state with $E \approx -0.317455\hbar^2/(ma^2)$ in the case that $\alpha = \hbar^2/(ma)$ is

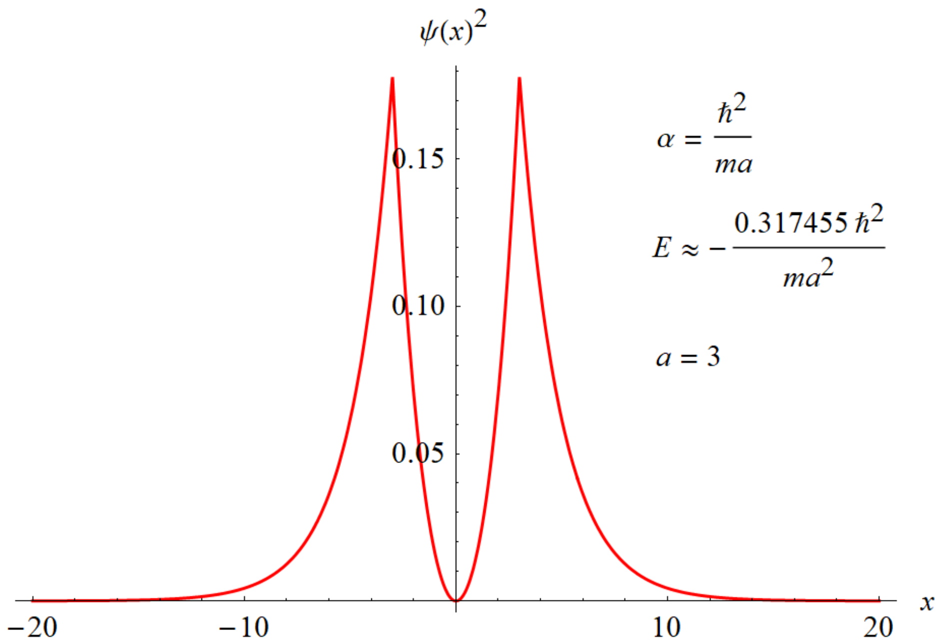
$$\boxed{\psi(x) = \begin{cases} \frac{1.62232}{\sqrt{a}} \exp\left(\frac{0.796812}{a}x\right) & \text{if } x \leq -a \\ \frac{0.413693}{\sqrt{a}} \left[-\exp\left(\frac{0.796812}{a}x\right) + \exp\left(-\frac{0.796812}{a}x\right)\right] & \text{if } -a \leq x \leq a \\ -\frac{1.62232}{\sqrt{a}} \exp\left(-\frac{0.796812}{a}x\right) & \text{if } x \geq a \end{cases}.}$$

Solving the ODE in t gives $\phi(t) = e^{-iEt/\hbar}$. The probability distribution for the particle's position at time t in this state is given by

$$|\psi(x)e^{-iEt/\hbar}|^2 = [\psi(x)e^{-iEt/\hbar}][\psi(x)e^{-iEt/\hbar}]^* = [\psi(x)e^{-iEt/\hbar}][\psi(x)e^{iEt/\hbar}] = [\psi(x)]^2.$$

Plots of $\psi(x)$ and $[\psi(x)]^2$ are shown below versus x for the special case that $a = 3$.





Notice that $\psi(x)$ is an odd function of x, that $\psi(x)$ is continuous across $x = \pm a$, that $d\psi/dx$ is discontinuous across $x = \pm a$, that $[\psi(x)]^2$ is an even function of x, that a particle in this state is most likely to be found near the delta-function wells, and that the distribution falls off exponentially away from the wells.

On the other hand, for $E \approx -0.614783\hbar^2/(ma^2)$, the corresponding eigenstate is

$$\psi(x) = \begin{cases} C_1 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x \le -a \\ C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar}x\right) + C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } -a \le x \le a \\ C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar}x\right) & \text{if } x \ge a \end{cases}$$

$$= \begin{cases} (C_3 + C_4 e^{2\kappa a}) \exp\left(\frac{1.10886}{a}x\right) & \text{if } x \le -a \\ C_3 \exp\left(\frac{1.10886}{a}x\right) + C_4 \exp\left(-\frac{1.10886}{a}x\right) & \text{if } -a \le x \le a \\ (C_3 e^{2\kappa a} + C_4) \exp\left(-\frac{1.10886}{a}x\right) & \text{if } x \ge a \end{cases}$$

$$= \begin{cases} \left[\left(\frac{\hbar^2\kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} + C_4 e^{2\kappa a}\right] \exp\left(\frac{1.10886}{a}x\right) & \text{if } x \le -a \\ \left(\frac{\hbar^2\kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} \exp\left(\frac{1.10886}{a}x\right) + C_4 \exp\left(-\frac{1.10886}{a}x\right) & \text{if } -a \le x \le a \\ \left[\left(\frac{\hbar^2\kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} e^{2\kappa a} + C_4\right] \exp\left(-\frac{1.10886}{a}x\right) & \text{if } x \ge a \end{cases}$$

$$= \begin{cases} C_4 \left(\kappa a e^{2\kappa a}\right) \exp\left(\frac{1.10886}{a}x\right) & \text{if } x \le -a \\ C_4 \left[(\kappa a - 1)e^{2\kappa a} \exp\left(\frac{1.10886}{a}x\right) + \exp\left(-\frac{1.10886}{a}x\right)\right] & \text{if } -a \le x \le a \\ C_4 \left[(\kappa a - 1)e^{4\kappa a} + 1\right] \exp\left(-\frac{1.10886}{a}x\right) & \text{if } x \ge a \end{cases}$$

$$= \begin{cases} 10.1863 C_4 \exp\left(\frac{1.10886}{a}x\right) & \text{if } x \le -a \\ C_4 \left[\exp\left(\frac{1.10886}{a}x\right) + \exp\left(-\frac{1.10886}{a}x\right)\right] & \text{if } -a \le x \le a \,. \\ 10.1863 C_4 \exp\left(-\frac{1.10886}{a}x\right) & \text{if } x \ge a \end{cases}$$

Evaluate the normalization constant C_4.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} [\psi(x)]^2 \, dx \\ &= \int_{-\infty}^{-a} [\psi(x)]^2 \, dx + \int_{-a}^{a} [\psi(x)]^2 \, dx + \int_{a}^{\infty} [\psi(x)]^2 \, dx \\ &= \int_{-\infty}^{-a} \left[10.1863 C_4 \exp\left(\frac{1.10886}{a}x\right)\right]^2 dx + \int_{-a}^{a} C_4^2 \left[\exp\left(\frac{1.10886}{a}x\right) + \exp\left(-\frac{1.10886}{a}x\right)\right]^2 dx \\ &\qquad + \int_{a}^{\infty} \left[10.1863 C_4 \exp\left(-\frac{1.10886}{a}x\right)\right]^2 dx \\ &= 22.3726 a C_4^2 \end{aligned}$$

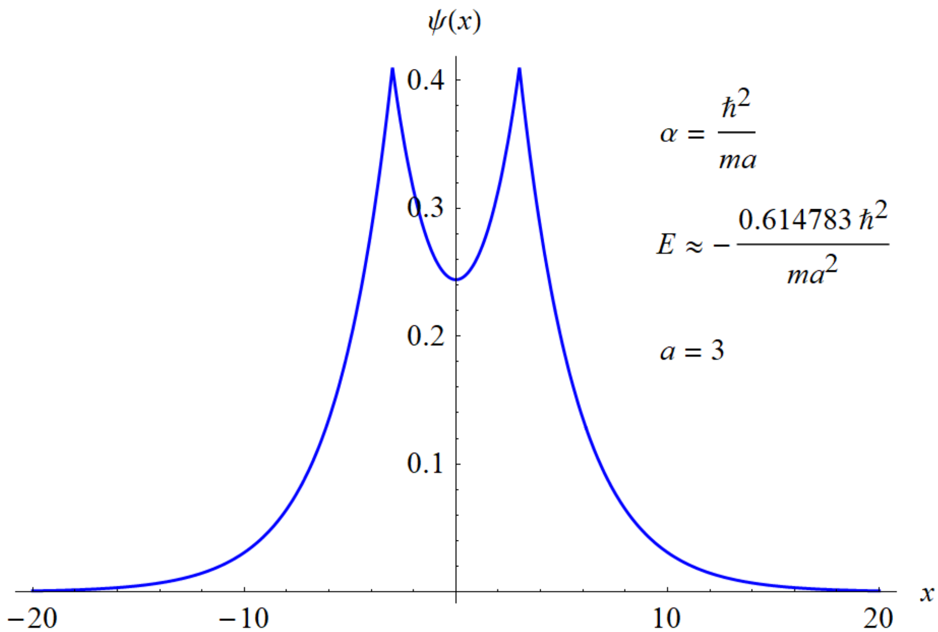
Solve for C_4.

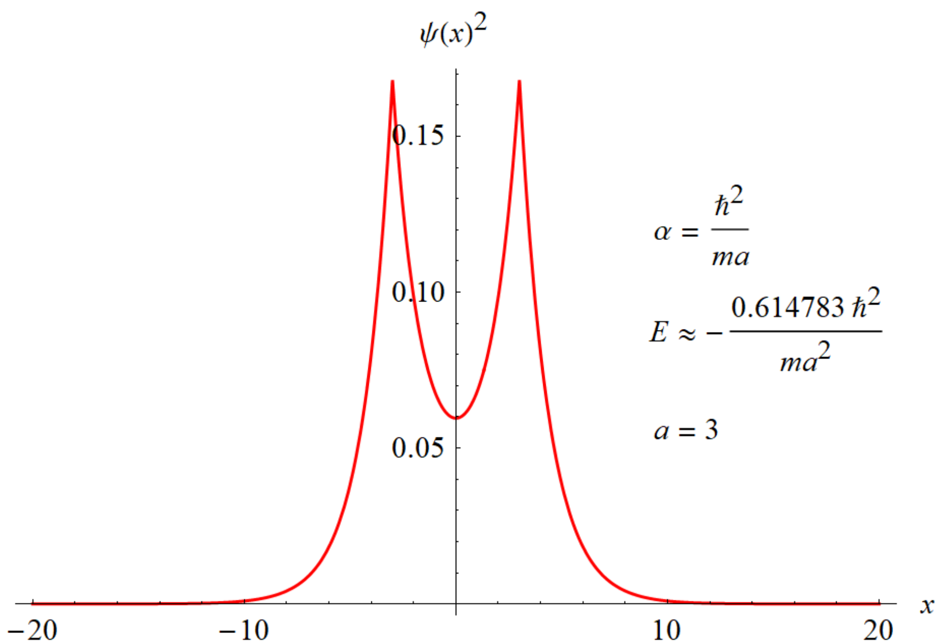
$$C_4 \approx \frac{0.211418}{\sqrt{a}}$$

Therefore, the bound state with $E \approx -0.614783\hbar^2/(ma^2)$ in the case that $\alpha = \hbar^2/(ma)$ is

$$\boxed{\psi(x) = \begin{cases} \frac{2.15357}{\sqrt{a}} \exp\left(\frac{1.10886}{a}x\right) & \text{if } x \le -a \\ \frac{0.211418}{\sqrt{a}} \left[\exp\left(\frac{1.10886}{a}x\right) + \exp\left(-\frac{1.10886}{a}x\right)\right] & \text{if } -a \le x \le a \\ \frac{2.15357}{\sqrt{a}} \exp\left(-\frac{1.10886}{a}x\right) & \text{if } x \ge a \end{cases}.}$$

Plots of $\psi(x)$ and $[\psi(x)]^2$ are shown below versus x for the special case that $a = 3$.





If $\alpha = \hbar^2/(4ma)$, then

$$1 = (4\kappa a - 1)^2 e^{4\kappa a} \quad \Rightarrow \quad \begin{cases} \kappa a \approx -0.628216 & \rightarrow \quad \cancel{\dfrac{\sqrt{-2mE}}{\hbar} a \approx -0.628216} \\ \kappa a = 0 & \rightarrow \quad \dfrac{\sqrt{-2mE}}{\hbar} a = 0 \quad \rightarrow \quad E = 0 \\ \kappa a \approx 0.369418 & \rightarrow \quad \dfrac{\sqrt{-2mE}}{\hbar} a \approx 0.369418 \quad \rightarrow \quad E \approx -\dfrac{0.0682347\hbar^2}{ma^2} \end{cases}.$$

This first result is nonsense because the square root of a positive number must be a positive number, so disregard it. Since there is one negative energy, there is one bound state for this value of α. In particular, the corresponding eigenstate to $E \approx -0.0682347\hbar^2/(ma^2)$ is

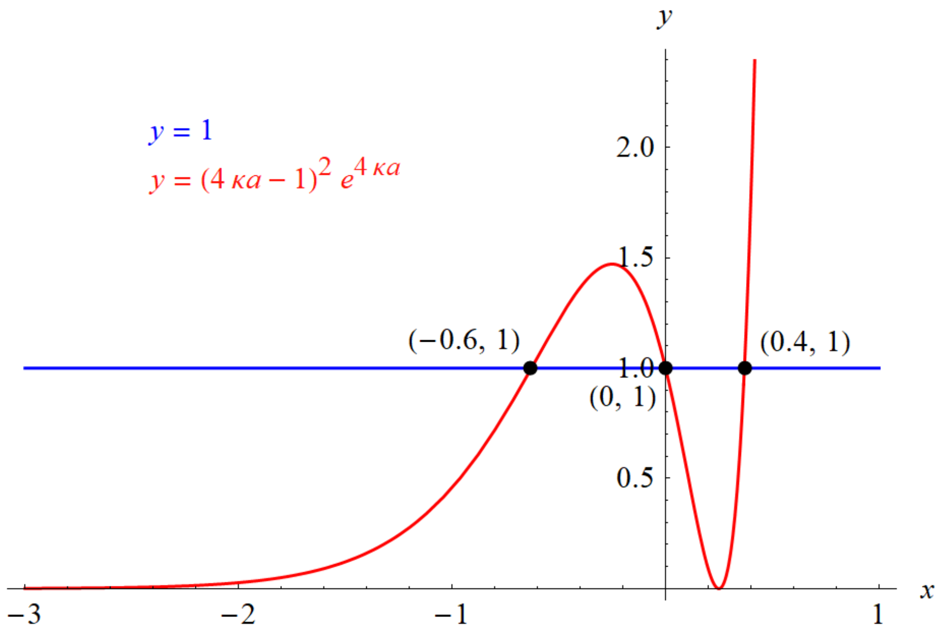
$$\psi(x) = \begin{cases} C_1 \exp\left(\frac{\sqrt{-2mE}}{\hbar} x\right) & \text{if } x \leq -a \\ C_3 \exp\left(\frac{\sqrt{-2mE}}{\hbar} x\right) + C_4 \exp\left(-\frac{\sqrt{-2mE}}{\hbar} x\right) & \text{if } -a \leq x \leq a \\ C_6 \exp\left(-\frac{\sqrt{-2mE}}{\hbar} x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} (C_3 + C_4 e^{2\kappa a}) \exp\left(\frac{0.369418}{a} x\right) & \text{if } x \leq -a \\ C_3 \exp\left(\frac{0.369418}{a} x\right) + C_4 \exp\left(-\frac{0.369418}{a} x\right) & \text{if } -a \leq x \leq a \\ (C_3 e^{2\kappa a} + C_4) \exp\left(-\frac{0.369418}{a} x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} \left[\left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} + C_4 e^{2\kappa a}\right] \exp\left(\frac{0.369418}{a} x\right) & \text{if } x \leq -a \\ \left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} \exp\left(\frac{0.369418}{a} x\right) + C_4 \exp\left(-\frac{0.369418}{a} x\right) & \text{if } -a \leq x \leq a \\ \left[\left(\frac{\hbar^2 \kappa}{m\alpha} - 1\right) C_4 e^{2\kappa a} e^{2\kappa a} + C_4\right] \exp\left(-\frac{0.369418}{a} x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} C_4 \left(4\kappa a e^{2\kappa a}\right) \exp\left(\frac{0.369418}{a} x\right) & \text{if } x \leq -a \\ C_4 \left[(4\kappa a - 1) e^{2\kappa a} \exp\left(\frac{0.369418}{a} x\right) + \exp\left(-\frac{0.369418}{a} x\right)\right] & \text{if } -a \leq x \leq a \\ C_4 \left[(4\kappa a - 1) e^{4\kappa a} + 1\right] \exp\left(-\frac{0.369418}{a} x\right) & \text{if } x \geq a \end{cases}$$

$$= \begin{cases} 3.09350 C_4 \exp\left(\frac{0.369418}{a} x\right) & \text{if } x \leq -a \\ C_4 \left[\exp\left(\frac{0.369418}{a} x\right) + \exp\left(-\frac{0.369418}{a} x\right)\right] & \text{if } -a \leq x \leq a \\ 3.09350 C_4 \exp\left(-\frac{0.369418}{a} x\right) & \text{if } x \geq a \end{cases}.$$



Evaluate the normalization constant C_4.

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} [\psi(x)]^2\,dx \\
&= \int_{-\infty}^{-a} [\psi(x)]^2\,dx + \int_{-a}^{a} [\psi(x)]^2\,dx + \int_{a}^{\infty} [\psi(x)]^2\,dx \\
&= \int_{-\infty}^{-a} \left[3.09350 C_4 \exp\left(\frac{0.369418}{a}x\right)\right]^2 dx + \int_{-a}^{a} C_4^2 \left[\exp\left(\frac{0.369418}{a}x\right) + \exp\left(-\frac{0.369418}{a}x\right)\right]^2 dx \\
&\qquad + \int_{a}^{\infty} \left[3.09350 C_4 \exp\left(-\frac{0.369418}{a}x\right)\right]^2 dx \\
&= 20.7480 a C_4^2
\end{aligned}
$$

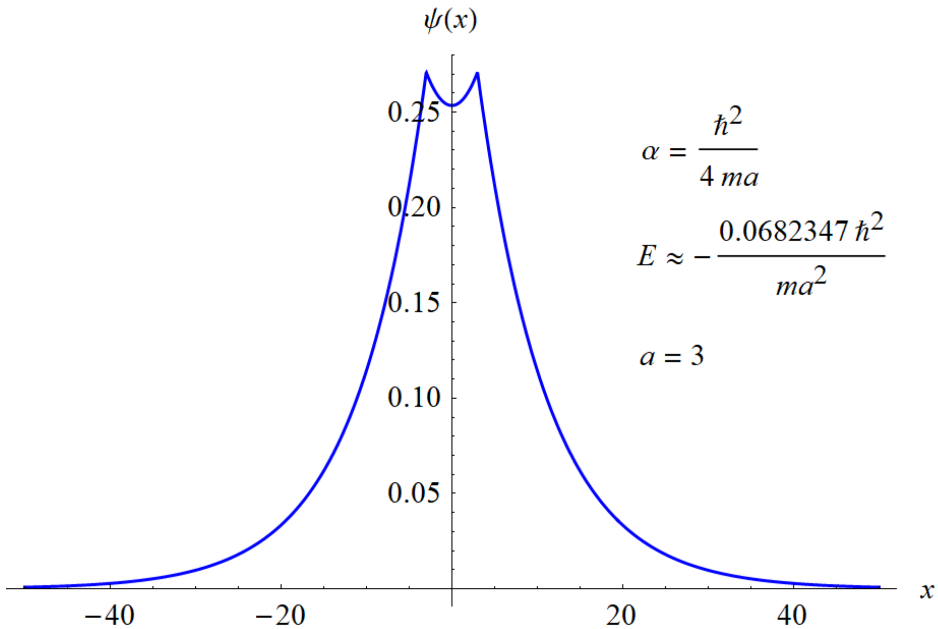
Solve for C_4.

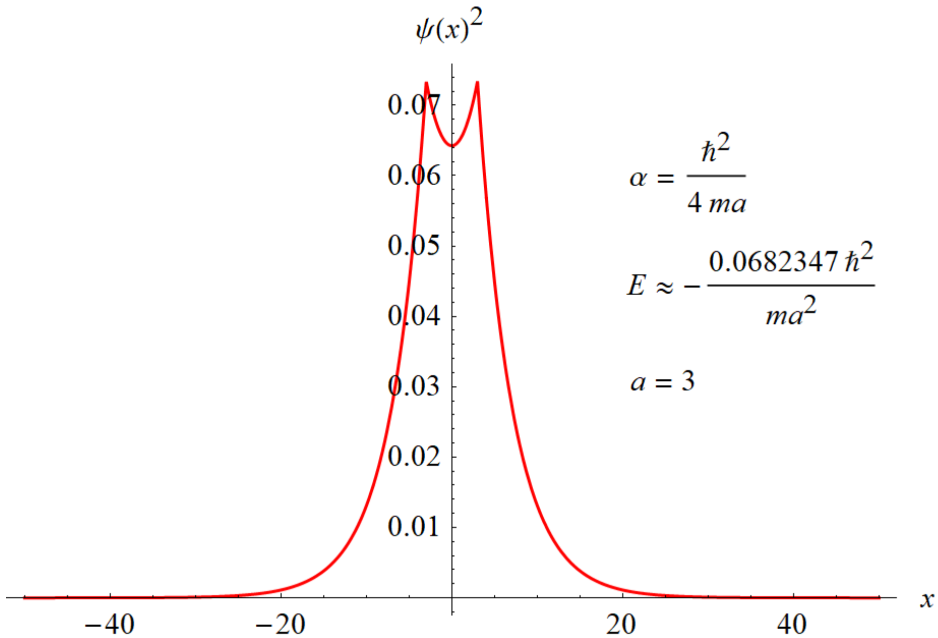
$$C_4 \approx \frac{0.219539}{\sqrt{a}}$$

Therefore, the bound state with $E \approx -0.0682347\hbar^2/(ma^2)$ in the case that $\alpha = \hbar^2/(4ma)$ is

$$
\boxed{\psi(x) = \begin{cases} \frac{0.679144}{\sqrt{a}} \exp\left(\frac{0.369418}{a}x\right) & \text{if } x \le -a \\ \frac{0.219539}{\sqrt{a}} \left[\exp\left(\frac{0.369418}{a}x\right) + \exp\left(-\frac{0.369418}{a}x\right)\right] & \text{if } -a \le x \le a \\ \frac{0.679144}{\sqrt{a}} \exp\left(-\frac{0.369418}{a}x\right) & \text{if } x \ge a \end{cases}.}
$$

Plots of $\psi(x)$ and $[\psi(x)]^2$ are shown below versus x for the special case that $a = 3$.





We found that if $\alpha = \hbar^2/(ma)$, then there are two bound states and that if $\alpha = \hbar^2/(4ma)$, then there is one bound state. Set $\alpha = \hbar^2/(Bma)$ in equation (2), the equation for the eigenvalues.

$$1 = (B\kappa a - 1)^2 e^{4\kappa a}$$

Graphing $y = 1$ and $y = (B\kappa a - 1)^2 e^{4\kappa a}$ versus κa for various values of B, we see there are two intersections for positive κa if $B < 2$ and only one intersection for positive κa if $B \geq 2$. Therefore,

$$\boxed{\begin{array}{l} \text{if } \alpha \leq \dfrac{\hbar^2}{2ma}, \text{ there is one bound state.} \\ \text{if } \alpha > \dfrac{\hbar^2}{2ma}, \text{ there are two bound states.} \end{array}}$$

To determine the bound state energies in the limiting cases, $a \to 0$ and $a \to \infty$, return to the equations involving C_1, C_3, C_4, and C_6.

$$\begin{cases} C_1 e^{-\kappa a} = C_3 e^{-\kappa a} + C_4 e^{\kappa a} \\ C_6 e^{-\kappa a} = C_3 e^{\kappa a} + C_4 e^{-\kappa a} \\ \dfrac{2m\alpha}{\hbar^2} C_1 e^{-\kappa a} = \kappa C_1 e^{-\kappa a} - \left(\kappa C_3 e^{-\kappa a} - \kappa C_4 e^{\kappa a}\right) \\ \dfrac{2m\alpha}{\hbar^2} C_6 e^{-\kappa a} = \left(\kappa C_3 e^{\kappa a} - \kappa C_4 e^{-\kappa a}\right) + \kappa C_6 e^{-\kappa a} \end{cases}$$

Write the four equations as follows for the first case.

$$\begin{cases} C_1 e^{-\kappa a} = C_3 e^{-\kappa a} + C_4 e^{\kappa a} \\ C_6 e^{-\kappa a} = C_3 e^{\kappa a} + C_4 e^{-\kappa a} \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) C_1 e^{-\kappa a} = -\left(\kappa C_3 e^{-\kappa a} - \kappa C_4 e^{\kappa a}\right) \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) C_6 e^{-\kappa a} = \left(\kappa C_3 e^{\kappa a} - \kappa C_4 e^{-\kappa a}\right) \end{cases}$$

Take the limit as $a \to 0$.

$$\begin{cases} C_1 = C_3 + C_4 \\ C_6 = C_3 + C_4 \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) C_1 = -\left(\kappa C_3 - \kappa C_4\right) \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) C_6 = \left(\kappa C_3 - \kappa C_4\right) \end{cases}$$

Since $C_1 = C_6$, these last two equations become

$$\left(\frac{2m\alpha}{\hbar^2} - \kappa\right) C_1 = -\left(\kappa C_3 - \kappa C_4\right) = \left(\kappa C_3 - \kappa C_4\right).$$

The only number equal to its negative is zero.

$$\left(\frac{2m\alpha}{\hbar^2} - \kappa\right) C_1 = 0$$

To avoid the trivial solution, insist that $C_1 \neq 0$. Then

$$\kappa = \frac{2m\alpha}{\hbar^2} \quad \to \quad \frac{\sqrt{-2mE}}{\hbar} = \frac{2m\alpha}{\hbar^2} \quad \to \quad \boxed{E = -\frac{2m\alpha^2}{\hbar^2}.}$$

Write the four equations the same way for the second case.

$$\begin{cases} C_1e^{-\kappa a} = C_3e^{-\kappa a} + C_4e^{\kappa a} \\ C_6e^{-\kappa a} = C_3e^{\kappa a} + C_4e^{-\kappa a} \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) C_1e^{-\kappa a} = -\left(\kappa C_3e^{-\kappa a} - \kappa C_4e^{\kappa a}\right) \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) C_6e^{-\kappa a} = \left(\kappa C_3e^{\kappa a} - \kappa C_4e^{-\kappa a}\right) \end{cases}$$

But then substitute these first two equations into the last two.

$$\begin{cases} \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) \left(C_3e^{-\kappa a} + C_4e^{\kappa a}\right) = -\left(\kappa C_3e^{-\kappa a} - \kappa C_4e^{\kappa a}\right) \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) \left(C_3e^{\kappa a} + C_4e^{-\kappa a}\right) = \left(\kappa C_3e^{\kappa a} - \kappa C_4e^{-\kappa a}\right) \end{cases}$$

Take the limit as $a \to \infty$.

$$\begin{cases} \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) (C_4e^{\kappa a}) = -(-\kappa C_4e^{\kappa a}) \\ \left(\dfrac{2m\alpha}{\hbar^2} - \kappa\right) (C_3e^{\kappa a}) = (\kappa C_3e^{\kappa a}) \end{cases}$$

Cancelling the common factors in both equations yields

$$\frac{2m\alpha}{\hbar^2} - \kappa = \kappa \quad \rightarrow \quad \kappa = \frac{m\alpha}{\hbar^2} \quad \rightarrow \quad \frac{\sqrt{-2mE}}{\hbar} = \frac{m\alpha}{\hbar^2} \quad \rightarrow \quad \boxed{E = -\frac{m\alpha^2}{2\hbar^2}.}$$

The single-well potential $V(x) = -\alpha\delta(x)$ has a bound state energy of

$$E = -\frac{m\alpha^2}{2\hbar^2}.$$

Plugging in 2α for α here gives the result for the limit of the double-well potential as $a \to 0$.