Problem 2.30

Normalize $\psi(x)$ in Equation 2.154, to determine the constants D and F.

Solution

Equation 2.154 gives the even bound states for the finite square well potential.

$$\psi(x) = \begin{cases} Fe^{\kappa x} & \text{if } x < -a \\ D\cos(lx) & \text{if } -a \leq x \leq a \\ Fe^{-\kappa x} & \text{if } x > a \end{cases} \tag{2.154}$$

Here

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} \quad \text{and} \quad l = \frac{\sqrt{2m(E+V_0)}}{\hbar}.$$

Continuity of $\psi(x)$ at $x = a$ yields

$$Fe^{-\kappa a} = D\cos(la), \tag{1}$$

and continuity of $d\psi/dx$ at $x = a$ yields

$$-\kappa Fe^{-\kappa a} = -lD\sin(la). \tag{2}$$

Solve equation (1) for F

$$F = De^{\kappa a}\cos(la)$$

and plug it into $\psi(x)$.

$$\psi(x) = \begin{cases} De^{\kappa a}\cos(la)e^{\kappa x} & \text{if } x < -a \\ D\cos(lx) & \text{if } -a \leq x \leq a \\ De^{\kappa a}\cos(la)e^{-\kappa x} & \text{if } x > a \end{cases}$$

D is arbitrary and chosen to have the value that makes the integral of $[\psi(x)]^2$ over the whole line 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} [\psi(x)]^2\,dx \\
&= \int_{-\infty}^{-a} [\psi(x)]^2\,dx + \int_{-a}^{a} [\psi(x)]^2\,dx + \int_{a}^{\infty} [\psi(x)]^2\,dx \\
&= \int_{-\infty}^{-a} [De^{\kappa a}\cos(la)e^{\kappa x}]^2\,dx + \int_{-a}^{a} [D\cos(lx)]^2\,dx + \int_{a}^{\infty} [De^{\kappa a}\cos(la)e^{-\kappa x}]^2\,dx \\
&= \int_{-\infty}^{-a} [D^2e^{2\kappa a}\cos^2(la)e^{2\kappa x}]\,dx + \int_{-a}^{a} [D^2\cos^2(lx)]\,dx + \int_{a}^{\infty} [D^2e^{2\kappa a}\cos^2(la)e^{-2\kappa x}]\,dx \\
&= D^2e^{2\kappa a}\cos^2(la)\int_{-\infty}^{-a} e^{2\kappa x}\,dx + 2D^2\int_0^a \frac{1}{2}[1+\cos(2lx)]\,dx + D^2e^{2\kappa a}\cos^2(la)\int_a^{\infty} e^{-2\kappa x}\,dx \\
&= D^2e^{2\kappa a}\cos^2(la)\left(\frac{1}{2\kappa}e^{2\kappa x}\bigg|_{-\infty}^{-a}\right) + D^2\left[x + \frac{1}{2l}\sin(2lx)\right]\bigg|_0^a + D^2e^{2\kappa a}\cos^2(la)\left(-\frac{1}{2\kappa}e^{-2\kappa x}\bigg|_a^{\infty}\right) \\
&= D^2e^{2\kappa a}\cos^2(la)\left(\frac{1}{2\kappa}e^{-2\kappa a}\right) + D^2\left[a + \frac{1}{2l}\sin(2la)\right] + D^2e^{2\kappa a}\cos^2(la)\left(\frac{1}{2\kappa}e^{-2\kappa a}\right) \\
&= D^2\left[\frac{1}{\kappa}\cos^2(la) + a + \frac{1}{2l}\sin(2la)\right]
\end{aligned}$$

Solve for D.

$$D = \frac{1}{\sqrt{\frac{1}{\kappa}\cos^2(la) + a + \frac{1}{2l}\sin(2la)}}$$

Divide the respective sides of equation (2) by those of equation (1).

$$\frac{-\kappa F e^{-\kappa a}}{F e^{-\kappa a}} = \frac{-lD\sin(la)}{D\cos(la)} \quad \rightarrow \quad \kappa = \frac{l\sin(la)}{\cos(la)} \quad \rightarrow \quad \frac{1}{l}\cos(la) = \frac{1}{\kappa}\sin(la)$$

Therefore, the normalization constant becomes

$$\begin{aligned} D &= \frac{1}{\sqrt{\frac{1}{\kappa}\cos^2(la) + a + \frac{1}{2l}[2\sin(la)\cos(la)]}} \\ &= \frac{1}{\sqrt{\frac{1}{\kappa}\cos^2(la) + a + \frac{1}{l}\sin(la)\cos(la)}} \\ &= \frac{1}{\sqrt{\frac{1}{\kappa}\cos^2(la) + a + \frac{1}{\kappa}\sin^2(la)}} \\ &= \frac{1}{\sqrt{\frac{1}{\kappa} + a}}, \end{aligned}$$

and

$$F = De^{\kappa a}\cos(la) = \frac{e^{\kappa a}\cos(la)}{\sqrt{\frac{1}{\kappa} + a}}.$$