Problem 2.40

A particle of mass m in the harmonic oscillator potential (Equation 2.44) starts out in the state

$$\Psi(x,0) = A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2},$$

for some constant A.

(a) Determine A and the coefficients c_n in the expansion of this state in terms of the stationary states of the harmonic oscillator.

(b) In a measurement of the particle's energy, what results could you get, and what are their probabilities? What is the expectation value of the energy?

(c) At a later time T the wave function is

$$\Psi(x,T) = B\left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2},$$

for some constant B. What is the smallest possible value of T?

Solution

Part (a)

Start by normalizing the initial wave function: Determine A by requiring the integral of $|\Psi(x,0)|^2$ over the whole line to be 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2\,dx \\
&= \int_{-\infty}^{\infty} A^2\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^4 \exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\
&= A^2\int_{-\infty}^{\infty}\left(1 - 8\sqrt{\frac{m\omega}{\hbar}}x + \frac{24m\omega}{\hbar}x^2 - 32\sqrt{\frac{m^3\omega^3}{\hbar^3}}x^3 + \frac{16m^2\omega^2}{\hbar^2}x^4\right)\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\
&= A^2\left[\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx - 8\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty}x\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx + \frac{24m\omega}{\hbar}\int_{-\infty}^{\infty}x^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx\right. \\
&\qquad \left. -32\sqrt{\frac{m^3\omega^3}{\hbar^3}}\int_{-\infty}^{\infty}x^3\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx + \frac{16m^2\omega^2}{\hbar^2}\int_{-\infty}^{\infty}x^4\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx\right]
\end{aligned}$$

The integrals with x and x^3 are zero because the integrands are odd and the integration intervals are symmetric. The other integrands are even; place factors of 2 in front of the rest and set the integration intervals to $(0,\infty)$.

$$1 = A^2\left[2\int_0^{\infty}\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx + \frac{48m\omega}{\hbar}\int_0^{\infty}x^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx + \frac{32m^2\omega^2}{\hbar^2}\int_0^{\infty}x^4\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx\right]$$

Write the integrands in the proper form and use the integration formulas inside the back cover of the textbook.

$$1 = A^2 \left\{ 2\int_0^\infty \exp\left[-\frac{x^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right] dx + \frac{48m\omega}{\hbar}\int_0^\infty x^2 \exp\left[-\frac{x^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right] dx \right.$$

$$\left. + \frac{32m^2\omega^2}{\hbar^2}\int_0^\infty x^4 \exp\left[-\frac{x^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right] dx \right\}$$

$$= A^2 \left[2\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right) + \frac{48m\omega}{\hbar}\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3 + \frac{32m^2\omega^2}{\hbar^2}\cdot\sqrt{\pi}\,\frac{4!}{2!}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^5 \right]$$

$$= A^2\left(\sqrt{\frac{\pi\hbar}{m\omega}} + 12\sqrt{\frac{\pi\hbar}{m\omega}} + 12\sqrt{\frac{\pi\hbar}{m\omega}}\right)$$

$$= 25A^2\sqrt{\frac{\pi\hbar}{m\omega}}$$

Solve for A.

$$A = \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Therefore, the initial wave function is

$$\Psi(x,0) = \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

$$= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(1 - 4\sqrt{\frac{m\omega}{\hbar}}x + \frac{4m\omega}{\hbar}x^2\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

$$= \left[\frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} - \frac{4}{5}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x + \frac{4m\omega}{5\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} x^2\right] \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

Since the polynomial in x is only up to x^2, only the first three eigenstates for the harmonic oscillator are needed. The final result of Problem 2.10 gives the first three terms of the general solution to Schrödinger's equation with $V(x,t) = (1/2)m\omega^2x^2$.

$$\Psi(x,t) = B_0\psi_0(x)e^{-iE_0t/\hbar} + B_1\psi_1(x)e^{-iE_1t/\hbar} + B_2\psi_2(x)e^{-iE_2t/\hbar} + \cdots$$

$$= B_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)e^{-i\omega t/2}$$

$$+ B_1\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)e^{-3i\omega t/2}$$

$$+ B_2\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right)e^{-5i\omega t/2} + \cdots$$

Set $t = 0$ and write it similar to the initial wave function.

$$\Psi(x,0) = B_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + B_1\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + B_2\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right)+\cdots$$

$$= \left[B_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} + B_1\sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x + \frac{B_2}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)+\cdots\right]\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

$$= \left[B_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} - \frac{B_2}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} + B_1\sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x + \frac{B_2}{\sqrt{2}}\frac{2m\omega}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}x^2+\cdots\right]\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

$$= \left[\left(B_0 - \frac{B_2}{\sqrt{2}}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} + B_1\sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x + B_2\frac{\sqrt{2}m\omega}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}x^2+\cdots\right]\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

Match the coefficients of the powers of x to get a system of equations for B_0, B_1, and B_2.

$$\left(B_0 - \frac{B_2}{\sqrt{2}}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} = \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$B_1\sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/4} = -\frac{4}{5}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/4}$$

$$B_2\frac{\sqrt{2}m\omega}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} = \frac{4m\omega}{5\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$B_n = 0, \quad n > 2$$

Solve it.

$$B_0 = \frac{3}{5}$$

$$B_1 = -\frac{2\sqrt{2}}{5}$$

$$B_2 = \frac{2\sqrt{2}}{5}$$

$$B_n = 0, \quad n > 2$$

Part (b)

Writing the general solution in terms of $\psi_0(x)$, $\psi_1(x)$, $\ldots$,

$$\Psi(x,t) = \frac{3}{5}\psi_0(x)e^{-iE_0t/\hbar} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-iE_1t/\hbar} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-iE_2t/\hbar},$$

we see that the probabilities of measuring $E_0 = \hbar\omega/2$, $E_1 = 3\hbar\omega/2$, and $E_2 = 5\hbar\omega/2$ are

$$P(E_0) = \left|\frac{3}{5}\right|^2 = \frac{9}{25} \quad\text{and}\quad P(E_1) = \left|-\frac{2\sqrt{2}}{5}\right|^2 = \frac{8}{25} \quad\text{and}\quad P(E_2) = \left|\frac{2\sqrt{2}}{5}\right|^2 = \frac{8}{25}.$$

Therefore, the expectation value of energy is

$$\begin{aligned}\langle E\rangle = \sum_n E_n P(E_n) = E_0P(E_0) + E_1P(E_1) + E_2P(E_2) &= \frac{1}{2}\hbar\omega\left(\frac{9}{25}\right) + \frac{3}{2}\hbar\omega\left(\frac{8}{25}\right) + \frac{5}{2}\hbar\omega\left(\frac{8}{25}\right)\\ &= \frac{73}{50}\hbar\omega\\ &= 1.46\hbar\omega.\end{aligned}$$

Part (c)

At $t = T$ the wave function becomes

$$\begin{aligned}\Psi(x,T) &= B\left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\ &= B\left(1 + 4\sqrt{\frac{m\omega}{\hbar}}x + \frac{4m\omega}{\hbar}x^2\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right),\end{aligned}$$

which is the same as the one initially except for the positive sign in front of the second term. Set $t = T$ in the general solution.

$$\begin{aligned}\Psi(x,T) &= \frac{3}{5}\psi_0(x)e^{-iE_0T/\hbar} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-iE_1T/\hbar} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-iE_2T/\hbar}\\ &= \frac{3}{5}\psi_0(x)e^{-i\omega T/2} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-3i\omega T/2} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-5i\omega T/2}\end{aligned}$$

Since the Schrödinger equation is linear and homogeneous, any constant multiple of a solution is also a solution. If this constant has a magnitude of 1, then the wave function will remain normalized. Include the constant e^{-irT}, then, where r is a real constant.

$$\begin{aligned}\Psi(x,T) &= \frac{3}{5}e^{-irT}\psi_0(x)e^{-i\omega T/2} - \frac{2\sqrt{2}}{5}e^{-irT}\psi_1(x)e^{-3i\omega T/2} + \frac{2\sqrt{2}}{5}e^{-irT}\psi_2(x)e^{-5i\omega T/2}\\ &= \frac{3}{5}\psi_0(x)e^{-i\left(r+\frac{\omega}{2}\right)T} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-i\left(r+\frac{3\omega}{2}\right)T} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-i\left(r+\frac{5\omega}{2}\right)T}\end{aligned}$$

The aim is to choose r in order to minimize T, which must satisfy

$$\left.\begin{aligned}\exp\left[-i\left(r+\frac{\omega}{2}\right)T\right] &= 1\\ \exp\left[-i\left(r+\frac{3\omega}{2}\right)T\right] &= -1\\ \exp\left[-i\left(r+\frac{5\omega}{2}\right)T\right] &= 1\end{aligned}\right\}.$$

There's only a solution if $r = -\omega/2 + 2\omega q$, where q is an integer. For any choice of r, T comes to the same value. Choosing $r = -5\omega/2$, for example, this system becomes

$$\left.\begin{aligned}e^{2i\omega T} &= 1\\ e^{i\omega T} &= -1\\ e^0 &= 1\end{aligned}\right\} \quad\rightarrow\quad \left.\begin{aligned}\cos 2\omega T + i\sin 2\omega T &= 1\\ \cos\omega T + i\sin\omega T &= -1\end{aligned}\right\} \quad\Rightarrow\quad \left.\begin{aligned}2\omega T &= 2p\pi\\ \omega T &= p\pi\end{aligned}\right\} \ (p \text{ odd}).$$

The smallest positive T occurs when the integer p is 1. Therefore, $T = \pi/\omega$.