Problem 2.42

In Problem 2.21 you analyzed the *stationary* gaussian free particle wave packet. Now solve the same problem for the *traveling* gaussian wave packet, starting with the initial wave function

$$\Psi(x,0) = Ae^{-ax^2}e^{ilx},$$

where l is a (real) constant. [*Suggestion:* In going from $\phi(k)$ to $\Psi(x,t)$, change variables to $u \equiv k - l$ before doing the integral.] *Partial answer:*

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\gamma}e^{-a(x-\hbar lt/m)^2/\gamma^2}e^{il(x-\hbar lt/2m)}$$

where $\gamma \equiv \sqrt{1 + 2ia\hbar t/m}$, as before. Notice that $\Psi(x,t)$ has the structure of a gaussian "envelope" modulating a traveling sinusoidal wave. What is the speed of the envelope? What is the speed of the traveling wave?

Solution

The governing equation for the wave function is the Schrödinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

For a free particle in particular, $V = 0$, and the equation simplifies to

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}.$$

Let there be a prescribed initial condition, $\Psi(x,0) = \Psi_0(x)$. Since the PDE is linear and $-\infty < x < \infty$, the Fourier transform can be applied to solve it. The Fourier transform of $\Psi(x,t)$ is defined here as

$$\mathcal{F}\{\Psi(x,t)\} = \tilde{\Psi}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\Psi(x,t)\,dx.$$

As a result, the partial derivatives of Ψ transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n\Psi}{\partial x^n}\right\} = (ik)^n\tilde{\Psi}(k,t)$$

$$\mathcal{F}\left\{\frac{\partial^n\Psi}{\partial t^n}\right\} = \frac{d^n\tilde{\Psi}}{dt^n}$$

Take the Fourier transform of both sides of the Schrödinger equation.

$$\mathcal{F}\left\{i\hbar\frac{\partial\Psi}{\partial t}\right\} = \mathcal{F}\left\{-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\right\}$$

Since the transform is a linear operator, the constants can be brought in front.

$$i\hbar\mathcal{F}\left\{\frac{\partial\Psi}{\partial t}\right\} = -\frac{\hbar^2}{2m}\mathcal{F}\left\{\frac{\partial^2\Psi}{\partial x^2}\right\}$$

Transform the derivatives.

$$i\hbar\frac{d\tilde{\Psi}}{dt} = -\frac{\hbar^2}{2m}(ik)^2\tilde{\Psi}(k,t)$$

By using the Fourier transform, Schrödinger's equation has become a first-order ODE. Solve it now.

$$\frac{\frac{d\tilde{\Psi}}{dt}}{\tilde{\Psi}} = -\frac{i\hbar}{2m}k^2$$

$$\frac{d}{dt}\ln\tilde{\Psi} = -\frac{i\hbar}{2m}k^2$$

$$\ln\tilde{\Psi} = -\frac{i\hbar}{2m}k^2 t + C(k)$$

$$\tilde{\Psi}(k,t) = B(k)\exp\left(-\frac{i\hbar}{2m}k^2 t\right)$$

To determine $B(k)$, transform the given initial condition,

$$\Psi(x,0) = \Psi_0(x) \quad \rightarrow \quad \mathcal{F}\{\Psi(x,0)\} = \mathcal{F}\{\Psi_0(x)\} \quad \rightarrow \quad \tilde{\Psi}(k,0) = \tilde{\Psi}_0(k),$$

and then apply it.

$$\tilde{\Psi}(k,0) = B(k) = \tilde{\Psi}_0(k)$$

Consequently, the transformed solution to Schrödinger's equation is

$$\tilde{\Psi}(k,t) = \tilde{\Psi}_0(k)\exp\left(-\frac{i\hbar}{2m}k^2 t\right).$$

Now take the inverse Fourier transform of both sides to get $\Psi(x,t)$.

$$\Psi(x,t) = \mathcal{F}^{-1}\left\{\tilde{\Psi}(k,t)\right\} = \mathcal{F}^{-1}\left\{\tilde{\Psi}_0(k)\exp\left(-\frac{i\hbar}{2m}k^2 t\right)\right\}$$

According to the convolution theorem for the Fourier transform,

$$\mathcal{F}^{-1}\left\{\tilde{f}(k)\tilde{g}(k)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

The inverse Fourier transform of $\tilde{\Psi}_0(k)$ is just $\Psi_0(x)$, and the inverse Fourier transform of the exponential function is

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\exp\left(-\frac{i\hbar}{2m}k^2 t\right)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\exp\left(-\frac{i\hbar}{2m}k^2 t\right)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{i\hbar}{2m}k^2 t + ikx\right)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar}{2m}t\left(k^2 - \frac{2mx}{\hbar t}k\right)\right]dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar}{2m}t\left(k^2 - \frac{2mx}{\hbar t}k + \frac{m^2x^2}{\hbar^2t^2}\right) + \frac{i\hbar}{2m}t\left(\frac{m^2x^2}{\hbar^2t^2}\right)\right]dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar t}{2m}\left(k - \frac{mx}{\hbar t}\right)^2\right]\exp\left(\frac{imx^2}{2\hbar t}\right)dk.
\end{aligned}$$

Make the following substitution.

$$u = k - \frac{mx}{\hbar t}$$

$$du = dk$$

Consequently,

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\exp\left(-\frac{i\hbar}{2m}k^2 t\right)\right\} &= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{imx^2}{2\hbar t}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{i\hbar t}{2m}u^2\right)du \\
&= \frac{2}{\sqrt{2\pi}}\exp\left(\frac{imx^2}{2\hbar t}\right)\int_0^{\infty}\exp\left[-\frac{u^2}{\left(\sqrt{\frac{2m}{i\hbar t}}\right)^2}\right]du \\
&= \frac{2}{\sqrt{2\pi}}\exp\left(\frac{imx^2}{2\hbar t}\right)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{2m}{i\hbar t}}}{2}\right) \\
&= \sqrt{\frac{m}{i\hbar t}}\exp\left(\frac{imx^2}{2\hbar t}\right).
\end{aligned}$$

By the convolution theorem, then,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_0(\xi)\sqrt{\frac{m}{i\hbar t}}\exp\left[\frac{im(x-\xi)^2}{2\hbar t}\right]d\xi.$$

Therefore, the solution to the Schrödinger equation for a free particle is

$$\boxed{\Psi(x,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right]\Psi_0(\xi)\,d\xi.}$$

In this problem

$$\Psi(x,0) = \Psi_0(x) = Ae^{-ax^2}e^{ilx}.$$

Part (a)

Start by normalizing the initial wave function: Require that the integral of $|\Psi(x,0)|^2$ over the whole line is 1 in order to determine A.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx \\
&= \int_{-\infty}^{\infty}(Ae^{-ax^2}e^{ilx})(Ae^{-ax^2}e^{-ilx})\,dx \\
&= \int_{-\infty}^{\infty}A^2e^{-2ax^2}\,dx \\
&= 2A^2\int_0^{\infty}\exp\left[-\frac{x^2}{\left(\frac{1}{\sqrt{2a}}\right)^2}\right]dx \\
&= 2A^2\cdot\sqrt{\pi}\left(\frac{\frac{1}{\sqrt{2a}}}{2}\right) \\
&= A^2\sqrt{\frac{\pi}{2a}}
\end{aligned}$$

Solve for A.

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

As a result, the initial wave function is

$$\Psi_0(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2} e^{ilx}.$$

Part (b)

Substitute the initial wave function into the boxed formula for $\Psi(x,t)$.

$$\begin{aligned}
\Psi(x,t) &= \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right] \Psi_0(\xi)\, d\xi \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right] \left(\frac{2a}{\pi}\right)^{1/4} e^{-a\xi^2} e^{il\xi}\, d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2 - a\xi^2 + il\xi\right] d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left(\frac{im}{2\hbar t}x^2 - i\frac{mx-\hbar lt}{\hbar t}\xi + \frac{im-2\hbar at}{2\hbar t}\xi^2\right) d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left(\frac{im}{2\hbar t}x^2\right) \exp\left\{\frac{im-2\hbar at}{2\hbar t}\left[\xi^2 - \frac{2i(mx-\hbar lt)}{im-2\hbar at}\xi\right]\right\} d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im}{2\hbar t}x^2\right) \int_{-\infty}^{\infty} \exp\left\{\frac{im-2\hbar at}{2\hbar t}\left[\xi - \frac{i(mx-\hbar lt)}{im-2\hbar at}\right]^2 - \frac{im-2\hbar at}{2\hbar t}\frac{i^2(mx-\hbar lt)^2}{(im-2\hbar at)^2}\right\} d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im}{2\hbar t}x^2\right) \int_{-\infty}^{\infty} \exp\left\{\frac{im-2\hbar at}{2\hbar t}\left[\xi - \frac{i(mx-\hbar lt)}{im-2\hbar at}\right]^2\right\} \exp\left[\frac{(mx-\hbar lt)^2}{2\hbar t(im-2\hbar at)}\right] d\xi
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
u &= \xi - \frac{i(mx-\hbar lt)}{im-2\hbar at} \\
du &= d\xi
\end{aligned}$$

As a result,

$$\begin{aligned}
\Psi(x,t) &= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{im}{2\hbar t}x^2 + \frac{(mx-\hbar lt)^2}{2\hbar t(im-2\hbar at)}\right] \int_{-\infty}^{\infty} \exp\left(\frac{im-2\hbar at}{2\hbar t}u^2\right) du \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{im(im-2\hbar at)x^2 + (mx-\hbar lt)^2}{2\hbar t(im-2\hbar at)}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{u^2}{\left(\sqrt{\frac{2\hbar t}{2\hbar at - im}}\right)^2}\right] du \\
&= 2\left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{-2i\hbar amtx^2 - 2m\hbar lxt + \hbar^2 l^2 t^2}{2\hbar t(im-2\hbar at)}\right] \int_0^{\infty} \exp\left[-\frac{u^2}{\left(\sqrt{\frac{2\hbar t}{2\hbar at - im}}\right)^2}\right] du.
\end{aligned}$$

Use the integration formula inside the back cover of the textbook and simplify.

$$
\begin{aligned}
\Psi(x,t) &= 2\left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{-2i\hbar amt}{2\hbar t(im-2\hbar at)}x^2\right]\exp\left[\frac{\hbar^2l^2t^2-2m\hbar lxt}{2\hbar t(im-2\hbar at)}\right]\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{2\hbar t}{2\hbar at-im}}}{2}\right) \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{i(2\hbar at-im)}}\exp\left(\frac{-iam}{im-2\hbar at}x^2\right)\exp\left(\frac{\frac{\hbar l^2 t}{2}-mlx}{im-2\hbar at}\right) \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(\frac{-a}{1+\frac{2i\hbar at}{m}}x^2\right)\exp\left(\frac{-i\frac{\hbar l^2t}{2m}+ilx}{1+\frac{2i\hbar at}{m}}\right) \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(\frac{-ax^2+\frac{2\hbar lta}{m}x-\frac{\hbar^2l^2t^2a}{m^2}}{1+\frac{2i\hbar at}{m}}\right)\exp\left(\frac{-i\frac{\hbar l^2t}{2m}+ilx-\frac{2\hbar lta}{m}x+\frac{\hbar^2l^2t^2a}{m^2}}{1+\frac{2i\hbar at}{m}}\right) \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x^2-\frac{2\hbar lt}{m}x+\frac{\hbar^2l^2t^2}{m^2}\right)}{1+\frac{2i\hbar at}{m}}\right]\exp\left[\frac{il\left(x-\frac{\hbar lt}{2m}\right)\left(1+\frac{2i\hbar at}{m}\right)}{1+\frac{2i\hbar at}{m}}\right] \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]
\end{aligned}
$$

Therefore, setting $\gamma=\sqrt{1+2i\hbar at/m}$,

$$
\Psi(x,t)=\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\gamma}e^{-a(x-\hbar lt/m)^2/\gamma^2}e^{il(x-\hbar lt/2m)}.
$$

This first exponential function, $e^{-a(x-\hbar lt/m)^2/\gamma^2}$, is a gaussian envelope, and this second exponential function, $e^{il(x-\hbar lt/2m)}$, is a sinusoidal wave. Their speeds are $\hbar l/m$ and $\hbar l/(2m)$, respectively.

Part (c)

Now that the wave function is known, the probability distribution for the particle's position at time t can be determined.

$$
\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)\Psi^*(x,t) \\
&= \left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\} \\
&\quad\times\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1-\frac{2i\hbar at}{m}}\right]\exp\left[-il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\} \\
&= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{\left(1+\frac{2i\hbar at}{m}\right)\left(1-\frac{2i\hbar at}{m}\right)}}\exp\left[-a\left(x-\frac{\hbar lt}{m}\right)^2\left(\frac{1}{1+\frac{2i\hbar at}{m}}+\frac{1}{1-\frac{2i\hbar at}{m}}\right)\right] \\
&= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left[-a\left(x-\frac{\hbar lt}{m}\right)^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]
\end{aligned}
$$

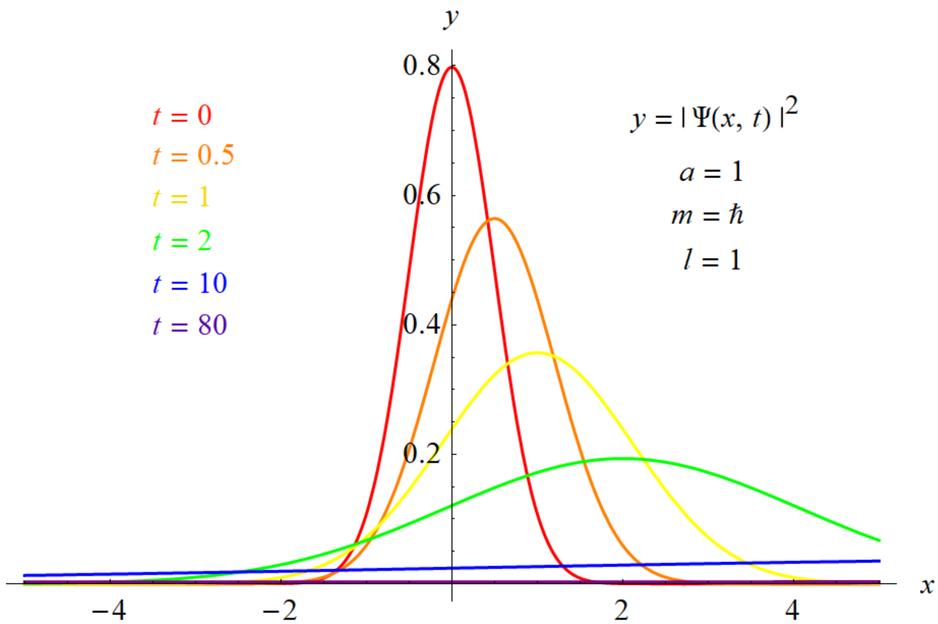
Therefore,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left[-2\left(\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\left(x-\frac{\hbar lt}{m}\right)^2\right],$$

or if we set $w = \sqrt{a/\left[1+(2\hbar at/m)^2\right]}$,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\,we^{-2w^2(x-\hbar lt/m)^2}.$$

Below is a plot of the probability distribution versus x at several times for the special case that $a = 1$ and $m = \hbar$ and $l = 1$.



Part (d)

Now that the probability distribution for the particle's position at time t is known, expectation values can be determined. Calculate the expectation value of x at time t.

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(x)\Psi(x,t)\,dx\\ &= \int_{-\infty}^{\infty}x|\Psi(x,t)|^2\,dx\\ &= \int_{-\infty}^{\infty}x\sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left[-2\left(\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\left(x-\frac{\hbar lt}{m}\right)^2\right]dx\\ &= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}x\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right]dx\end{aligned}$$

Make the following substitution.

$$\begin{aligned}v &= x-\frac{\hbar lt}{m} \quad\to\quad x = v+\frac{\hbar lt}{m}\\ dv &= dx\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}\left(v+\frac{\hbar lt}{m}\right)\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv \\
&= m\sqrt{\frac{2a}{\pi(m^2+4\hbar^2a^2t^2)}}\left[\underbrace{\int_{-\infty}^{\infty}v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv}_{=\,0}+\frac{\hbar lt}{m}\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right] \\
&= \hbar lt\sqrt{\frac{2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv \\
&= 2\hbar lt\sqrt{\frac{2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{0}^{\infty}\exp\left[-\frac{v^2}{\left(\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}\right)^2}\right]dv \\
&= 2\hbar lt\sqrt{\frac{2a}{\pi(m^2+4\hbar^2a^2t^2)}}\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right) \\
&= \frac{\hbar lt}{m}.
\end{aligned}$$

Now calculate the expectation value of x^2 at time t.

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}x^2|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty}x^2\sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left[-2\left(\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\left(x-\frac{\hbar lt}{m}\right)^2\right]dx \\
&= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}x^2\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right]dx
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
v &= x-\frac{\hbar lt}{m} \quad\rightarrow\quad x=v+\frac{\hbar lt}{m} \\
dv &= dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle x^2\rangle &= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}\left(v+\frac{\hbar lt}{m}\right)^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv \\
&= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}\left(v^2+\frac{2\hbar lt}{m}v+\frac{\hbar^2l^2t^2}{m^2}\right)\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv \\
&= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\left[\int_{-\infty}^{\infty}v^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv+\frac{2\hbar lt}{m}\overbrace{\int_{-\infty}^{\infty}v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv}^{=0}\right. \\
&\qquad\left.+\frac{\hbar^2l^2t^2}{m^2}\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right] \\
&= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\left[\int_{-\infty}^{\infty}v^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv+\frac{\hbar^2l^2t^2}{m^2}\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right] \\
&= 2\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\left\{\int_0^{\infty}v^2\exp\left[-\frac{v^2}{\left(\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}\right)^2}\right]dv+\frac{\hbar^2l^2t^2}{m^2}\int_0^{\infty}\exp\left[-\frac{v^2}{\left(\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}\right)^2}\right]dv\right\} \\
&= 2\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\left[\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)^3+\frac{\hbar^2l^2t^2}{m^2}\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)\right] \\
&= \frac{m^2+4\hbar^2a^2t^2}{4m^2a}+\frac{\hbar^2l^2t^2}{m^2}.
\end{aligned}$$

The standard deviation in x at time t is then

$$\sigma_x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\left(\frac{m^2+4\hbar^2a^2t^2}{4m^2a}+\frac{\hbar^2l^2t^2}{m^2}\right)-\left(\frac{\hbar lt}{m}\right)^2}=\sqrt{\frac{m^2+4\hbar^2a^2t^2}{4m^2a}}.$$

According to Ehrenfest's theorem,

$$\begin{aligned}
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \\
&= m\frac{d}{dt}\left(\frac{\hbar lt}{m}\right) \\
&= \hbar l.
\end{aligned}$$

Check this result by calculating the expectation value of p at time t.

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial \Psi}{\partial x}\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} \left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1-\frac{2i\hbar at}{m}}\right]\exp\left[-il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\} \\
&\qquad \times \frac{\partial}{\partial x}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}dx \\
&= -i\hbar\int_{-\infty}^{\infty} \left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1-\frac{2i\hbar at}{m}}\right]\exp\left[-il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\} \\
&\qquad \times \left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\left(\frac{il-2ax}{1+\frac{2i\hbar at}{m}}\right)\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}dx \\
&= -i\hbar\int_{-\infty}^{\infty}\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}(il-2ax)\exp\left[-a\left(x-\frac{\hbar lt}{m}\right)^2\left(\frac{1}{1-\frac{2i\hbar at}{m}}+\frac{1}{1+\frac{2i\hbar at}{m}}\right)\right]dx \\
&= -i\hbar\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}\int_{-\infty}^{\infty}(il-2ax)\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right]dx \\
&= -i\hbar\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}\left\{il\int_{-\infty}^{\infty}\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right]dx\right. \\
&\qquad \left. -2a\int_{-\infty}^{\infty}x\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right]dx\right\}
\end{aligned}$$

Make the following substitution in both integrals.

$$\begin{aligned}
v &= x-\frac{\hbar lt}{m} \quad \rightarrow \quad x = v+\frac{\hbar lt}{m} \\
dv &= dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle p \rangle = -i\hbar\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}&\left\{il\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right. \\
&\left. -2a\int_{-\infty}^{\infty}\left(v+\frac{\hbar lt}{m}\right)\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right\}.
\end{aligned}$$

Proceed to evaluate the integrals.

$$\begin{aligned}
\langle p\rangle &= -i\hbar\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}\left\{il\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv - 2a\overbrace{\int_{-\infty}^{\infty}v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv}^{=0}\right.\\
&\qquad\left.-\frac{2\hbar alt}{m}\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right\}\\
&= -i\hbar\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}\left[\left(il-\frac{2\hbar alt}{m}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right]\\
&= -i\hbar\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^3}}\left[2il\left(1+\frac{2i\hbar at}{m}\right)\int_{0}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right]\\
&= \hbar l\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)}}\left[2\int_{0}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right]\\
&= 2\hbar l\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_0^\infty\exp\left[-\frac{v^2}{\left(\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}\right)^2}\right]dv\\
&= 2\hbar l\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)\\
&= \hbar l
\end{aligned}$$

This confirms Ehrenfest's theorem. Finally, calculate the expectation value of p^2 at time t.

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi(x,t)\,dx\\
&= -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x,t)\frac{\partial^2\Psi}{\partial x^2}\,dx\\
&= -\hbar^2\int_{-\infty}^{\infty}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1-\frac{2i\hbar at}{m}}\right]\exp\left[-il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}\\
&\qquad\times\frac{\partial^2}{\partial x^2}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}dx\\
&= -\hbar^2\int_{-\infty}^{\infty}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1-\frac{2i\hbar at}{m}}\right]\exp\left[-il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}\\
&\qquad\times\frac{\partial}{\partial x}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\left(\frac{il-2ax}{1+\frac{2i\hbar at}{m}}\right)\exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}dx
\end{aligned}$$

Evaluate the derivative and simplify the result.

$$\langle p^2\rangle = -\hbar^2 \int_{-\infty}^{\infty} \left\{ \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}} \exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1-\frac{2i\hbar at}{m}}\right] \exp\left[-il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\}$$

$$\times \left\{ \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{\left(1+\frac{2i\hbar at}{m}\right)^3}} \left(-\frac{l^2+2a+\frac{4i\hbar a^2 t}{m}+4ilax-4a^2x^2}{1+\frac{2i\hbar at}{m}}\right)\right.$$

$$\left.\times \exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right] \exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right]\right\} dx$$

$$= \hbar^2 \sqrt{\frac{2a}{\pi}} \frac{1}{\sqrt{\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}} \int_{-\infty}^{\infty} \left[\left(l^2+2a+\frac{4i\hbar a^2 t}{m}\right)+4ilax-4a^2x^2\right]$$

$$\times \exp\left[-a\left(x-\frac{\hbar lt}{m}\right)^2\left(\frac{1}{1-\frac{2i\hbar at}{m}}+\frac{1}{1+\frac{2i\hbar at}{m}}\right)\right] dx$$

$$= \hbar^2 \sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}} \int_{-\infty}^{\infty} \left[\left(l^2+2a+\frac{4i\hbar a^2 t}{m}\right)+4ilax-4a^2x^2\right]$$

$$\times \exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right] dx$$

$$= \hbar^2 \sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}} \left\{\left(l^2+2a+\frac{4i\hbar a^2 t}{m}\right)\int_{-\infty}^{\infty} \exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right] dx\right.$$

$$+4ila\int_{-\infty}^{\infty} x\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right] dx$$

$$\left.-4a^2\int_{-\infty}^{\infty} x^2\exp\left[-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\left(x-\frac{\hbar lt}{m}\right)^2\right] dx\right\}$$

Make the following substitution in all the integrals.

$$v = x-\frac{\hbar lt}{m} \quad\rightarrow\quad x = v+\frac{\hbar lt}{m}$$
$$dv = dx$$

Consequently,

$$\langle p^2\rangle = \hbar^2 \sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}} \left[\left(l^2+2a+\frac{4i\hbar a^2 t}{m}\right)\int_{-\infty}^{\infty} \exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right) dv\right.$$

$$+4ila\int_{-\infty}^{\infty} \left(v+\frac{\hbar lt}{m}\right)\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right) dv$$

$$\left.-4a^2\int_{-\infty}^{\infty} \left(v+\frac{\hbar lt}{m}\right)^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right) dv\right].$$

Proceed to evaluate the integrals.

$$\begin{aligned}
\langle p^2\rangle &= \hbar^2\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}}\left[\left(l^2+2a+\frac{4i\hbar a^2t}{m}+\frac{4i\hbar al^2t}{m}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right.\\
&\qquad + \cancel{4ila\int_{-\infty}^{\infty}v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv}\\
&\qquad \left. -4a^2\int_{-\infty}^{\infty}\left(v^2+\frac{2\hbar lt}{m}v+\frac{\hbar^2l^2t^2}{m^2}\right)\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right]\\
&= \hbar^2\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}}\left[\left(l^2+2a+\frac{4i\hbar a^2t}{m}+\frac{4i\hbar al^2t}{m}-\frac{4\hbar^2a^2l^2t^2}{m^2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right.\\
&\qquad -4a^2\int_{-\infty}^{\infty}v^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\\
&\qquad \left. \cancel{-\frac{8\hbar a^2lt}{m}\int_{-\infty}^{\infty}v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv}\right]\\
&= \hbar^2\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}}\left[2\left(1+\frac{2i\hbar at}{m}\right)\left(l^2+2a+\frac{2i\hbar al^2t}{m}\right)\int_0^{\infty}\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right.\\
&\qquad \left. -8a^2\int_0^{\infty}v^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)dv\right]\\
&= \hbar^2\sqrt{\frac{2a}{\pi\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)^5}}\left\{2l^2\left(1+\frac{2i\hbar at}{m}\right)\left(\frac{2a}{l^2}+1+\frac{2i\hbar at}{m}\right)\int_0^{\infty}\exp\left[-\frac{v^2}{\left(\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}\right)^2}\right]dv\right.\\
&\qquad \left. -8a^2\int_0^{\infty}v^2\exp\left[-\frac{v^2}{\left(\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}\right)^2}\right]dv\right\}\\
&= \hbar^2\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)\left(1+\frac{2i\hbar at}{m}\right)^4}}\left[2l^2\left(1+\frac{2i\hbar at}{m}\right)\left(\frac{2a}{l^2}+1+\frac{2i\hbar at}{m}\right)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)\right.\\
&\qquad \left. -8a^2\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)^3\right]\\
&= \frac{\hbar^2}{\left(1+\frac{2i\hbar at}{m}\right)^2}\left[l^2\left(1+\frac{2i\hbar at}{m}\right)\left(\frac{2a}{l^2}+1+\frac{2i\hbar at}{m}\right)-2a^2\left(\frac{m^2+4\hbar^2a^2t^2}{2m^2a}\right)\right]\\
&= \frac{\hbar^2}{\left(1+\frac{2i\hbar at}{m}\right)^2}\left[(a+l^2)\left(1+\frac{2i\hbar at}{m}\right)^2\right]\\
&= \hbar^2(a+l^2)
\end{aligned}$$

The standard deviation in p at time t is then

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{(\hbar^2 a + \hbar^2 l^2) - (\hbar l)^2} = \hbar\sqrt{a}.$$

Part (e)

The uncertainty product is

$$\sigma_x \sigma_p = \sqrt{\frac{m^2 + 4\hbar^2 a^2 t^2}{4m^2 a}}(\hbar\sqrt{a}) = \frac{\hbar}{2}\sqrt{1 + \frac{4\hbar^2 a^2 t^2}{m^2}},$$

which is consistent with Heisenberg's uncertainty principle ($\sigma_x \sigma_p \geq \hbar/2$) for all t. This system comes closest to the limit at $t = 0$.