Problem 2.46

Imagine a bead of mass m that slides frictionlessly around a circular wire ring of circumference L. (This is just like a free particle, except that $\psi(x+L) = \psi(x)$.) Find the stationary states (with appropriate normalization) and the corresponding allowed energies. Note that there are (with one exception) *two* independent solutions for each energy E_n—corresponding to clockwise and counter-clockwise circulation; call them $\psi_n^+(x)$ and $\psi_n^-(x)$. How do you account for this degeneracy, in view of the theorem in Problem 2.44 (why does the theorem fail, in this case)?

Solution

Despite the fact that a ring lies in a two-dimensional plane, only one coordinate x is needed to describe the bead's position on it. The aim is then to solve the Schrödinger equation in one dimension for the wave function $\Psi(x,t)$.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

Because the ring is frictionless, the bead moves freely on it and $V(x,t) = 0$.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

The wave function and its slope must be L-periodic because x and $x+L$ are at the same location on the ring.

$$\begin{aligned}\Psi(x,t) &= \Psi(x+L,t)\\ \frac{\partial\Psi}{\partial x}(x,t) &= \frac{\partial\Psi}{\partial x}(x+L,t)\end{aligned}$$

These periodic boundary conditions are different from the usual Dirichlet boundary conditions ($\Psi \to 0$ as $x \to \pm\infty$) and are the reason why the degeneracy theorem in Problem 2.44 fails. To introduce symmetry and simplify the forthcoming algebra, these periodic conditions will be used with x set to $-L/2$.

$$\begin{aligned}\Psi\left(-\frac{L}{2},t\right) &= \Psi\left(\frac{L}{2},t\right)\\ \frac{\partial\Psi}{\partial x}\left(-\frac{L}{2},t\right) &= \frac{\partial\Psi}{\partial x}\left(\frac{L}{2},t\right)\end{aligned}$$

Since we want to know information about the stationary states and their corresponding energies, the method of separation of variables is opted for. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] \quad \to \quad i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t)$$

and its associated boundary conditions.

$$\begin{aligned}\Psi\left(-\frac{L}{2},t\right) = \Psi\left(\frac{L}{2},t\right) \quad &\to \quad \psi\left(-\frac{L}{2}\right)\phi(t) = \psi\left(\frac{L}{2}\right)\phi(t) \quad \to \quad \psi\left(-\frac{L}{2}\right) = \psi\left(\frac{L}{2}\right)\\ \frac{\partial\Psi}{\partial x}\left(-\frac{L}{2},t\right) = \frac{\partial\Psi}{\partial x}\left(\frac{L}{2},t\right) \quad &\to \quad \psi'\left(-\frac{L}{2}\right)\phi(t) = \psi'\left(\frac{L}{2}\right)\phi(t) \quad \to \quad \psi'\left(-\frac{L}{2}\right) = \psi'\left(\frac{L}{2}\right)\end{aligned}$$

Divide both sides of the PDE by $\psi(x)\phi(t)$ to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} = E$$

As a result of using the method of separation of variables, the Schrödinger equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions are satisfied are called the eigenvalues (or eigenenergies in this context), and the nontrivial solutions associated with them are called the eigenfunctions (or eigenstates in this context). The ODE in x is known as the time-independent Schrödinger equation (TISE) and can be written as

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi, \quad -\frac{L}{2} < x < \frac{L}{2}.$$

Check to see if there are negative eigenvalues: $E = -\gamma^2$.

$$\frac{d^2\psi}{dx^2} = \frac{2m\gamma^2}{\hbar^2}\psi$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\psi(x) = C_1\cosh\frac{\sqrt{2m}\gamma x}{\hbar} + C_2\sinh\frac{\sqrt{2m}\gamma x}{\hbar}$$

Apply the periodic boundary conditions.

$$\psi\left(-\frac{L}{2}\right) = \psi\left(\frac{L}{2}\right): \quad C_1\cosh\frac{\sqrt{2m}\gamma L}{2\hbar} - C_2\sinh\frac{\sqrt{2m}\gamma L}{2\hbar} = C_1\cosh\frac{\sqrt{2m}\gamma L}{2\hbar} + C_2\sinh\frac{\sqrt{2m}\gamma L}{2\hbar}$$

$$\begin{aligned} \psi'\left(-\frac{L}{2}\right) = \psi'\left(\frac{L}{2}\right): \quad & -\frac{\sqrt{2m}C_1\gamma}{\hbar}\sinh\frac{\sqrt{2m}\gamma L}{2\hbar} + \frac{\sqrt{2m}\gamma C_2}{\hbar}\cosh\frac{\sqrt{2m}\gamma L}{2\hbar} \\ &= \frac{\sqrt{2m}C_1\gamma}{\hbar}\sinh\frac{\sqrt{2m}\gamma L}{2\hbar} + \frac{\sqrt{2m}\gamma C_2}{\hbar}\cosh\frac{\sqrt{2m}\gamma L}{2\hbar} \end{aligned}$$

The hyperbolic cosines cancel, leaving

$$\left.\begin{aligned} 2C_2\sinh\frac{\sqrt{2m}\gamma L}{2\hbar} &= 0 \\ 2\frac{\sqrt{2m}C_1\gamma}{\hbar}\sinh\frac{\sqrt{2m}\gamma L}{2\hbar} &= 0 \end{aligned}\right\}.$$

There are no nonzero values of γ for which the hyperbolic sine vanishes, so $C_1 = 0$ and $C_2 = 0$. This leads to the trivial solution, $\psi(x) = 0$, which means there are no negative eigenvalues. Now check to see if zero is an eigenvalue: $E = 0$.

$$\frac{d^2\psi}{dx^2} = 0$$

The general solution is a straight line.

$$\psi(x) = C_3 x + C_4$$

Apply the periodic boundary conditions.

$$\begin{aligned} \psi\left(-\frac{L}{2}\right) = \psi\left(\frac{L}{2}\right) &: \quad -\frac{C_3 L}{2} + C_4 = \frac{C_3 L}{2} + C_4 \\ \psi'\left(-\frac{L}{2}\right) = \psi'\left(\frac{L}{2}\right) &: \quad C_3 = C_3 \end{aligned}$$

Solving this first equation yields $C_3 = 0$.

$$\psi(x) = C_4$$

C_4 is arbitrary and is chosen so that the integral of $[\psi(x)]^2$ over the ring's circumference is 1.

$$1 = \int_{-L/2}^{L/2} [\psi(x)]^2 \, dx \quad \Rightarrow \quad C_4 = \frac{1}{\sqrt{L}}$$

Solving the ODE in t with $E = 0$ yields $\phi(t) = e^{-iEt/\hbar} = 1$. Now check to see if there are positive eigenvalues: $E = \mu^2$.

$$\frac{d^2\psi}{dx^2} = -\frac{2m\mu^2}{\hbar^2}\psi$$

The general solution can be written in terms of sine and cosine.

$$\psi(x) = C_5 \cos\frac{\sqrt{2m}\mu x}{\hbar} + C_6 \sin\frac{\sqrt{2m}\mu x}{\hbar}$$

Apply the periodic boundary conditions.

$$\begin{aligned} \psi\left(-\frac{L}{2}\right) = \psi\left(\frac{L}{2}\right) &: \quad C_5 \cos\frac{\sqrt{2m}\mu L}{2\hbar} - C_6 \sin\frac{\sqrt{2m}\mu L}{2\hbar} = C_5 \cos\frac{\sqrt{2m}\mu L}{2\hbar} + C_6 \sin\frac{\sqrt{2m}\mu L}{2\hbar} \\ \psi'\left(-\frac{L}{2}\right) = \psi'\left(\frac{L}{2}\right) &: \quad \frac{\sqrt{2m}C_5\mu}{\hbar}\sin\frac{\sqrt{2m}\mu L}{2\hbar} + \frac{\sqrt{2m}\mu C_6}{\hbar}\cos\frac{\sqrt{2m}\mu L}{2\hbar} \\ &\qquad = -\frac{\sqrt{2m}C_5\mu}{\hbar}\sin\frac{\sqrt{2m}\mu L}{2\hbar} + \frac{\sqrt{2m}\mu C_6}{\hbar}\cos\frac{\sqrt{2m}\mu L}{2\hbar} \end{aligned}$$

The cosines cancel, leaving

$$\left.\begin{aligned} 2C_6 \sin\frac{\sqrt{2m}\mu L}{2\hbar} &= 0 \\ 2\frac{\sqrt{2m}C_5\mu}{\hbar}\sin\frac{\sqrt{2m}\mu L}{2\hbar} &= 0 \end{aligned}\right\}.$$

If we insist that $C_5 \neq 0$ and $C_6 \neq 0$, then both equations can be satisfied if

$$\begin{aligned} \sin \frac{\sqrt{2m}\mu L}{2\hbar} &= 0 \\ \frac{\sqrt{2m}\mu L}{2\hbar} &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= \sqrt{\frac{2}{m}} \frac{n\pi\hbar}{L}. \end{aligned}$$

Therefore, since $E = \mu^2$, the positive eigenvalues are

$$E_n = \frac{2n^2\pi^2\hbar^2}{mL^2}, \quad n = 1, 2, \ldots.$$

Only positive integers are taken for n because $n = 0$ leads to the zero eigenvalue, which was already considered, and negative integers lead to redundant values for E. The eigenfunctions associated with these eigenvalues are

$$\begin{aligned} \psi(x) &= C_5 \cos \frac{\sqrt{2m}\mu x}{\hbar} + C_6 \sin \frac{\sqrt{2m}\mu x}{\hbar} \\ &= C_5 \cos \frac{2n\pi x}{L} + C_6 \sin \frac{2n\pi x}{L}. \end{aligned}$$

Since there are two arbitrary constants, C_5 and C_6, and neither one can be written in terms of the other, there are two linearly independent solutions to the TISE for each energy E_n. Normalize these solutions now.

$$\begin{aligned} 1 &= \int_{-L/2}^{L/2} \left(C_5 \cos \frac{2n\pi x}{L} \right)^2 dx \qquad \Rightarrow \qquad C_5 = \sqrt{\frac{2}{L}} \\ 1 &= \int_{-L/2}^{L/2} \left(C_6 \sin \frac{2n\pi x}{L} \right)^2 dx \qquad \Rightarrow \qquad C_6 = \sqrt{\frac{2}{L}} \end{aligned}$$

Solving the ODE in t with $E = E_n$ yields $\phi(t) = e^{-iE_n t/\hbar}$. Therefore, the stationary states are

$$\begin{aligned} \Psi_0(x,t) &= \psi_0(x)\phi_0(t) = \frac{1}{\sqrt{L}}(1) = \frac{1}{\sqrt{L}} \\ \Psi_n(x,t) &= \sqrt{\frac{2}{L}} \cos \frac{2n\pi x}{L} e^{-iE_n t/\hbar} = \sqrt{\frac{2}{L}} \exp\left(-i \frac{2n^2\pi^2\hbar}{mL^2} t\right) \cos \frac{2n\pi x}{L} \\ \Psi_n(x,t) &= \sqrt{\frac{2}{L}} \sin \frac{2n\pi x}{L} e^{-iE_n t/\hbar} = \sqrt{\frac{2}{L}} \exp\left(-i \frac{2n^2\pi^2\hbar}{mL^2} t\right) \sin \frac{2n\pi x}{L}, \end{aligned}$$

and the allowed energies are $E = 0$ and $E_n = 2n^2\pi^2\hbar^2/(mL^2)$ with $n = 1, 2, \ldots$. According to the principle of superposition, the general solution to the Schrödinger equation is a linear combination of these stationary states.

$$\Psi(x,t) = A_0 \frac{1}{\sqrt{L}} + \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{L}} \exp\left(-i \frac{2n^2\pi^2\hbar}{mL^2} t\right) \cos \frac{2n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sqrt{\frac{2}{L}} \exp\left(-i \frac{2n^2\pi^2\hbar}{mL^2} t\right) \sin \frac{2n\pi x}{L}$$

With a given initial condition, $\Psi(x,0) = \Psi_0(x)$, the coefficients could be determined.

Note that with a different choice of constants, the eigenfunctions for the positive eigenvalues can be written in terms of complex exponential functions.

$$\begin{aligned}
\psi(x) &= C_5 \cos \frac{2n\pi x}{L} + C_6 \sin \frac{2n\pi x}{L} \\
&= C_5 \left(\frac{e^{2in\pi x/L} + e^{-2in\pi x/L}}{2} \right) + C_6 \left(\frac{e^{2in\pi x/L} - e^{-2in\pi x/L}}{2i} \right) \\
&= \left(\frac{C_5}{2} + \frac{C_6}{2i} \right) e^{2in\pi x/L} + \left(\frac{C_5}{2} - \frac{C_6}{2i} \right) e^{-2in\pi x/L} \\
&= C_7 e^{2in\pi x/L} + C_8 e^{-2in\pi x/L}
\end{aligned}$$

Normalizing these linearly independent solutions results in

$$1 = \int_{-L/2}^{L/2} \left(C_7 e^{2in\pi x/L} \right)^2 dx = C_7^2 L \qquad \Rightarrow \qquad C_7 = \frac{1}{\sqrt{L}}$$

$$1 = \int_{-L/2}^{L/2} \left(C_8 e^{-2in\pi x/L} \right)^2 dx = C_8^2 L \qquad \Rightarrow \qquad C_8 = \frac{1}{\sqrt{L}}.$$

The stationary states are then

$$\Psi_n(x,t) = \frac{1}{\sqrt{L}} e^{2in\pi x/L} e^{-iE_n t/\hbar} = \frac{1}{\sqrt{L}} \exp \left[i \left(\frac{2n\pi x}{L} - \frac{E_n t}{\hbar} \right) \right] = \frac{1}{\sqrt{L}} \exp \left[\frac{2in\pi}{L} \left(x - \frac{E_n L}{2n\pi\hbar} t \right) \right]$$

$$\Psi_n(x,t) = \frac{1}{\sqrt{L}} e^{-2in\pi x/L} e^{-iE_n t/\hbar} = \frac{1}{\sqrt{L}} \exp \left[-i \left(\frac{2n\pi x}{L} + \frac{E_n t}{\hbar} \right) \right] = \frac{1}{\sqrt{L}} \exp \left[-\frac{2in\pi}{L} \left(x + \frac{E_n L}{2n\pi\hbar} t \right) \right],$$

which can be interpreted as plane waves travelling in the direction of increasing x and decreasing x, respectively, with speed

$$\frac{E_n L}{2n\pi\hbar} = \frac{n\pi\hbar}{mL}.$$