Problem 2.49

(a) Show that

$$\Psi(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]$$

satisfies the time-*dependent* Schrödinger equation for the harmonic oscillator potential (Equation 2.44). Here x_0 is any real constant with the dimensions of length.[59]

(b) Find $|\Psi(x,t)|^2$, and describe the motion of the wave packet.

(c) Compute $\langle x\rangle$ and $\langle p\rangle$, and check that Ehrenfest's theorem (Equation 1.38) is satisfied.

Solution

Part (a)

Here the goal is merely to verify that

$$\Psi(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]$$

is a solution to the Schrödinger equation with a harmonic oscillator potential.

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\Psi(x,t)$$

Calculate the first derivative with respect to t.

$$\begin{aligned}\frac{\partial\Psi}{\partial t}&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]\\&\qquad\times\frac{\partial}{\partial t}\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]\\&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]\\&\qquad\times\left[-\frac{m\omega}{2\hbar}\left(\frac{x_0^2}{2}\left(-2i\omega e^{-2i\omega t}\right)+\frac{i\hbar}{m}-2x_0x(-i\omega)e^{-i\omega t}\right)\right]\\&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]\\&\qquad\times\left(-\frac{i\omega}{2}-\frac{im\omega^2xx_0e^{-i\omega t}}{\hbar}+\frac{im\omega^2x_0^2e^{-2i\omega t}}{2\hbar}\right)\end{aligned}$$

[59] This rare example of an exact closed-form solution to the time-dependent Schrödinger equation was discovered by Schrödinger himself, in 1926. One way to obtain it is explored in Problem 6.30. For a discussion of this and related problems see W. van Dijk, et al., *Am. J. Phys.* **82**, 955 (2014).

Calculate the first derivative with respect to x.

$$\begin{aligned}
\frac{\partial \Psi}{\partial x} &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad \times \frac{\partial}{\partial x}\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad \times \left[-\frac{m\omega}{2\hbar}\left(2x - 2x_0 e^{-i\omega t}\right)\right] \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right]\left[-\frac{m\omega}{\hbar}\left(x - x_0 e^{-i\omega t}\right)\right]
\end{aligned}$$

Calculate the second derivative with respect to x.

$$\begin{aligned}
\frac{\partial^2 \Psi}{\partial x^2} &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left\{\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right]\left[-\frac{m\omega}{\hbar}\left(x - x_0 e^{-i\omega t}\right)\right]^2\right. \\
&\qquad \left. + \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right]\left(-\frac{m\omega}{\hbar}\right)\right\} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad \times \left\{\left[-\frac{m\omega}{\hbar}\left(x - x_0 e^{-i\omega t}\right)\right]^2 + \left(-\frac{m\omega}{\hbar}\right)\right\} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad \times \left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2 x^2}{\hbar^2} - \frac{2m^2\omega^2 x x_0 e^{-i\omega t}}{\hbar^2} + \frac{m^2\omega^2 x_0^2 e^{-2i\omega t}}{\hbar^2}\right)
\end{aligned}$$

Evaluate the left side of Schrödinger's equation.

$$\begin{aligned}
i\hbar\frac{\partial \Psi}{\partial t} &= i\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad \times \left(-\frac{i\omega}{2} - \frac{im\omega^2 x x_0 e^{-i\omega t}}{\hbar} + \frac{im\omega^2 x_0^2 e^{-2i\omega t}}{2\hbar}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad \times \left(\frac{\hbar\omega}{2} + m\omega^2 x x_0 e^{-i\omega t} - \frac{m\omega^2 x_0^2 e^{-2i\omega t}}{2}\right)
\end{aligned}$$

Evaluate the right side of Schrödinger's equation.

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi(x,t) &= -\frac{\hbar^2}{2m}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right] \\
&\quad\times\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2x^2}{\hbar^2} - \frac{2m^2\omega^2xx_0e^{-i\omega t}}{\hbar^2} + \frac{m^2\omega^2x_0^2e^{-2i\omega t}}{\hbar^2}\right) \\
&\quad+\frac{1}{2}m\omega^2x^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right] \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right] \\
&\quad\times\left[-\frac{\hbar^2}{2m}\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2x^2}{\hbar^2} - \frac{2m^2\omega^2xx_0e^{-i\omega t}}{\hbar^2} + \frac{m^2\omega^2x_0^2e^{-2i\omega t}}{\hbar^2}\right) + \frac{1}{2}m\omega^2x^2\right] \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right] \\
&\quad\times\left(\frac{\hbar\omega}{2} - \cancel{\frac{m\omega^2x^2}{2}} + m\omega^2xx_0e^{-i\omega t} - \frac{m\omega^2x_0^2e^{-2i\omega t}}{2} + \cancel{\frac{m\omega^2x^2}{2}}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right] \\
&\quad\times\left(\frac{\hbar\omega}{2} + m\omega^2xx_0e^{-i\omega t} - \frac{m\omega^2x_0^2e^{-2i\omega t}}{2}\right)
\end{aligned}$$

Because both sides of the Schrödinger equation evaluate to the same function, the formula for $\Psi(x,t)$ is indeed a solution.

Part (b)

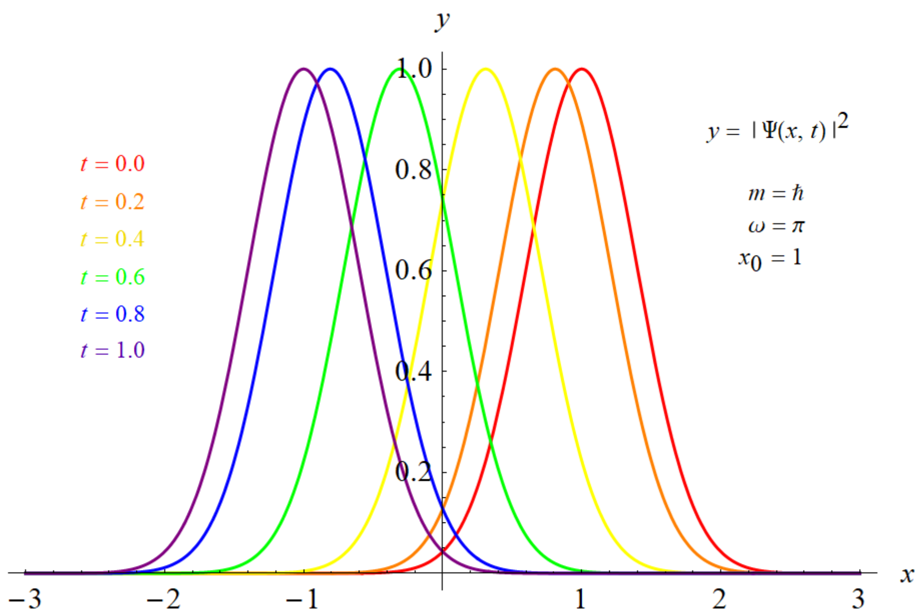
The probability distribution for the particle's position at time t is given by

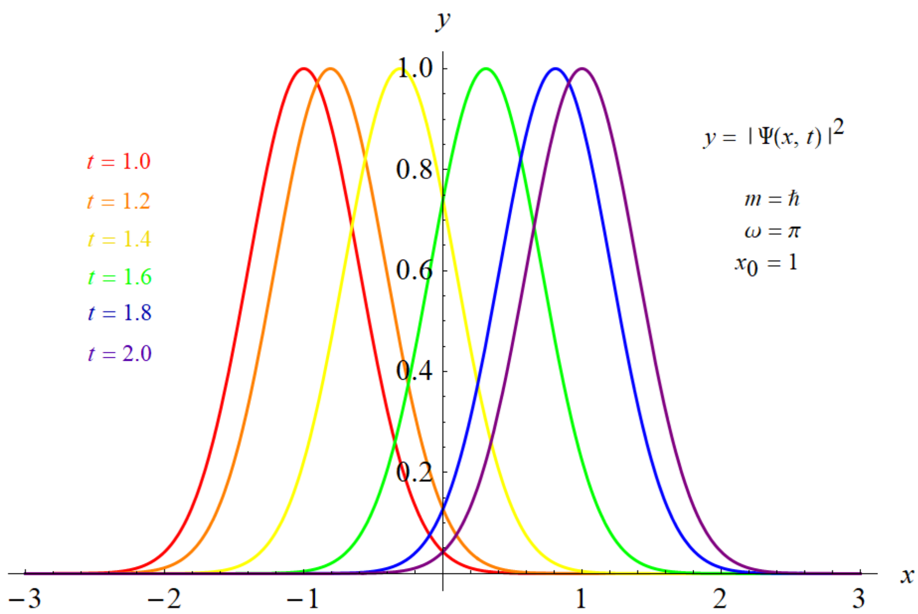
$$\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)\Psi^*(x,t) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right] \\
&\quad\times\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{2i\omega t}\right) - \frac{i\hbar t}{m} - 2x_0xe^{i\omega t}\right)\right] \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2 + x_0^2 + \frac{x_0^2}{2}(e^{2i\omega t} + e^{-2i\omega t}) - 2xx_0(e^{i\omega t} + e^{-i\omega t})\right)\right] \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2 + x_0^2 + \frac{x_0^2}{2}(2\cos 2\omega t) - 2xx_0(2\cos\omega t)\right)\right] \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2 + x_0^2(1+\cos 2\omega t) - 4xx_0\cos\omega t\right)\right].
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
|\Psi(x,t)|^2 &= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{2\hbar}\left(2x^2 + x_0^2(2\cos^2\omega t) - 4xx_0\cos\omega t\right)\right] \\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{\hbar}\left(x^2 - 2xx_0\cos\omega t + x_0^2\cos^2\omega t\right)\right] \\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{\hbar}\left(x - x_0\cos\omega t\right)^2\right]
\end{aligned}$$

Below are plots of $|\Psi(x,t)|^2$ versus x for several values of t with $m = \hbar$, $\omega = \pi$, and $x_0 = 1$.





Part (c)

Calculate the expectation value of x at time t.

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{\hbar}(x-x_0\cos\omega t)^2\right]dx
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
v &= x - x_0\cos\omega t \quad \rightarrow \quad x = v + x_0\cos\omega t \\
dv &= dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}(v + x_0\cos\omega t)\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\left[\underbrace{\int_{-\infty}^{\infty} v\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv}_{=0} + x_0\cos\omega t\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right] \\
&= 2x_0\sqrt{\frac{m\omega}{\pi\hbar}}\cos\omega t\int_0^{\infty}\exp\left[-\frac{v^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dv \\
&= \left(2x_0\sqrt{\frac{m\omega}{\pi\hbar}}\cos\omega t\right)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right) \\
&= x_0\cos\omega t.
\end{aligned}$$

Now calculate the expectation value of x^2.

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x^2\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{\hbar}(x-x_0\cos\omega t)^2\right]dx
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
v &= x - x_0\cos\omega t \quad \rightarrow \quad x = v + x_0\cos\omega t \\
dv &= dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle x^2\rangle &= \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}(v+x_0\cos\omega t)^2\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}(v^2+2vx_0\cos\omega t+x_0^2\cos^2\omega t)\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\left[\int_{-\infty}^{\infty}v^2\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv+2x_0\cos\omega t\underbrace{\int_{-\infty}^{\infty}v\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv}_{=\,0}+x_0^2\cos^2\omega t\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right] \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\left\{2\int_0^{\infty}v^2\exp\left[-\frac{v^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dv+2x_0^2\cos^2\omega t\int_0^{\infty}\exp\left[-\frac{v^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dv\right\} \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\left[2\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3+(2x_0^2\cos^2\omega t)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)\right] \\
&= \frac{\hbar}{2m\omega}+x_0^2\cos^2\omega t.
\end{aligned}$$

The standard deviation in x is then

$$\sigma_x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\left(\frac{\hbar}{2m\omega}+x_0^2\cos^2\omega t\right)-x_0^2\cos^2\omega t}=\sqrt{\frac{\hbar}{2m\omega}}.$$

According to Ehrenfest's theorem, the expectation value of p at time t is

$$\langle p\rangle=m\frac{d\langle x\rangle}{dt}=m\frac{d}{dt}(x_0\cos\omega t)=-m\omega x_0\sin\omega t,\ \text{and}\quad \frac{d\langle p\rangle}{dt}=-m\omega^2x_0\cos\omega t=-m\omega^2\langle x\rangle=\left\langle-\frac{d}{dx}\left(\frac{1}{2}m\omega^2x^2\right)\right\rangle.$$

Check this first result by calculating the expectation value of p at time t directly.

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi^*(x,t)\frac{\partial\Psi}{\partial x}\,dx \\
&= -i\hbar\int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{2i\omega t}\right)-\frac{i\hbar t}{m}-2x_0xe^{i\omega t}\right)\right] \\
&\qquad\times\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]\left[-\frac{m\omega}{\hbar}\left(x-x_0e^{-i\omega t}\right)\right] \\
&= -i\hbar\int_{-\infty}^{\infty}|\Psi(x,t)|^2\left[-\frac{m\omega}{\hbar}\left(x-x_0e^{-i\omega t}\right)\right]dx \\
&= -i\hbar\int_{-\infty}^{\infty}\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{\hbar}(x-x_0\cos\omega t)^2\right]\left[-\frac{m\omega}{\hbar}\left(x-x_0e^{-i\omega t}\right)\right]dx \\
&= im\omega\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(x-x_0e^{-i\omega t}\right)\exp\left[-\frac{m\omega}{\hbar}(x-x_0\cos\omega t)^2\right]dx
\end{aligned}$$

Make the following substitution.

$$v = x - x_0 \cos\omega t \quad \rightarrow \quad x = v + x_0 \cos\omega t$$
$$dv = dx$$

Consequently,

$$\begin{aligned}
\langle p \rangle &= im\omega\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(v + x_0\cos\omega t - x_0 e^{-i\omega t}\right)\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv \\
&= im\omega\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(v + \cancel{x_0\cos\omega t} - \cancel{x_0\cos\omega t} + ix_0\sin\omega t\right)\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv \\
&= im\omega\sqrt{\frac{m\omega}{\pi\hbar}}\left[\underbrace{\int_{-\infty}^{\infty} v\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv}_{=0} + ix_0\sin\omega t\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right] \\
&= -m\omega x_0\sqrt{\frac{m\omega}{\pi\hbar}}\sin\omega t\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv \\
&= -m\omega x_0\sqrt{\frac{m\omega}{\pi\hbar}}\sin\omega t\left(\sqrt{\frac{\pi\hbar}{m\omega}}\right) \\
&= -m\omega x_0\sin\omega t,
\end{aligned}$$

which confirms Ehrenfest's theorem. Finally, calculate the expectation value of p^2 at time t.

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi(x,t)\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x,t)\frac{\partial^2\Psi}{\partial x^2}\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{2i\omega t}\right) - \frac{i\hbar t}{m} - 2x_0 x e^{i\omega t}\right)\right] \\
&\qquad\times\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0 x e^{-i\omega t}\right)\right] \\
&\qquad\qquad\times\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2 x^2}{\hbar^2} - \frac{2m^2\omega^2 x x_0 e^{-i\omega t}}{\hbar^2} + \frac{m^2\omega^2 x_0^2 e^{-2i\omega t}}{\hbar^2}\right)dx \\
&= -\hbar^2\int_{-\infty}^{\infty}|\Psi(x,t)|^2\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2 x^2}{\hbar^2} - \frac{2m^2\omega^2 x x_0 e^{-i\omega t}}{\hbar^2} + \frac{m^2\omega^2 x_0^2 e^{-2i\omega t}}{\hbar^2}\right)dx \\
&= -\hbar^2\int_{-\infty}^{\infty}\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{\hbar}\left(x - x_0\cos\omega t\right)^2\right]\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2 x^2}{\hbar^2} - \frac{2m^2\omega^2 x x_0 e^{-i\omega t}}{\hbar^2} + \frac{m^2\omega^2 x_0^2 e^{-2i\omega t}}{\hbar^2}\right)dx
\end{aligned}$$

Make the following substitution.

$$v = x - x_0 \cos\omega t \quad \rightarrow \quad x = v + x_0 \cos\omega t$$
$$dv = dx$$

Consequently,

$$\begin{aligned}
\langle p^2 \rangle = -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)&\left[\frac{m^2\omega^2(v + x_0\cos\omega t)^2}{\hbar^2} - \frac{2m^2\omega^2(v + x_0\cos\omega t)x_0 e^{-i\omega t}}{\hbar^2}\right. \\
&\left. + \frac{m^2\omega^2 x_0^2 e^{-2i\omega t}}{\hbar^2} - \frac{m\omega}{\hbar}\right]dv.
\end{aligned}$$

Expand the quantity in square brackets and proceed to evaluate the integral.

$$\begin{aligned}
\langle p^2\rangle &= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)\left[\frac{m^2\omega^2}{\hbar^2}v^2+\left(\frac{2m^2\omega^2x_0\cos\omega t}{\hbar^2}-\frac{2m^2\omega^2x_0e^{-i\omega t}}{\hbar^2}\right)v\right.\\
&\qquad\left.+\frac{m^2\omega^2x_0^2e^{-2i\omega t}}{\hbar^2}-\frac{2m^2\omega^2x_0^2e^{-i\omega t}\cos\omega t}{\hbar^2}+\frac{m^2\omega^2x_0^2\cos^2\omega t}{\hbar^2}-\frac{m\omega}{\hbar}\right]\\
&= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{m^2\omega^2}{\hbar^2}\int_{-\infty}^{\infty}v^2\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv+\left(\frac{2m^2\omega^2x_0\cos\omega t}{\hbar^2}-\frac{2m^2\omega^2x_0e^{-i\omega t}}{\hbar^2}\right)\overbrace{\int_{-\infty}^{\infty}v\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv}^{=0}\right.\\
&\qquad\left.+\left(\frac{m^2\omega^2x_0^2e^{-2i\omega t}}{\hbar^2}-\frac{2m^2\omega^2x_0^2e^{-i\omega t}\cos\omega t}{\hbar^2}+\frac{m^2\omega^2x_0^2\cos^2\omega t}{\hbar^2}-\frac{m\omega}{\hbar}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right]\\
&= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{2m^2\omega^2}{\hbar^2}\int_0^{\infty}v^2\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right.\\
&\qquad\left.+\frac{2m^2\omega^2x_0^2}{\hbar^2}\left(e^{-2i\omega t}-2e^{-i\omega t}\cos\omega t+\cos^2\omega t-\frac{\hbar}{m\omega x_0^2}\right)\int_0^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right]\\
&= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\left\{\frac{2m^2\omega^2}{\hbar^2}\int_0^{\infty}v^2\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right.\\
&\qquad+\frac{2m^2\omega^2x_0^2}{\hbar^2}\left[\cos2\omega t-\cancel{i\sin2\omega t}-2(\cos\omega t-\cancel{i\sin\omega t})\cos\omega t+\cos^2\omega t-\frac{\hbar}{m\omega x_0^2}\right]\\
&\qquad\left.\times\int_0^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right\}\\
&= -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{2m^2\omega^2}{\hbar^2}\int_0^{\infty}v^2\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv+\frac{2m^2\omega^2x_0^2}{\hbar^2}\left(-1+\cos^2\omega t-\frac{\hbar}{m\omega x_0^2}\right)\int_0^{\infty}\exp\left(-\frac{m\omega}{\hbar}v^2\right)dv\right]\\
&= -\sqrt{\frac{m\omega}{\pi\hbar}}\left\{2m^2\omega^2\int_0^{\infty}v^2\exp\left[-\frac{v^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dv-2m^2\omega^2x_0^2\left(\sin^2\omega t+\frac{\hbar}{m\omega x_0^2}\right)\int_0^{\infty}\exp\left[-\frac{v^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dv\right\}\\
&= -\sqrt{\frac{m\omega}{\pi\hbar}}\left[2m^2\omega^2\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3-2m^2\omega^2x_0^2\left(\sin^2\omega t+\frac{\hbar}{m\omega x_0^2}\right)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)\right]\\
&= -2m^2\omega^2\cdot\frac{\hbar}{4m\omega}+m^2\omega^2x_0^2\left(\sin^2\omega t+\frac{\hbar}{m\omega x_0^2}\right)\\
&= \frac{\hbar m\omega}{2}+m^2\omega^2x_0^2\sin^2\omega t
\end{aligned}$$

The standard deviation in p is then

$$\sigma_p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\left(\frac{\hbar m\omega}{2}+m^2\omega^2x_0^2\sin^2\omega t\right)-m^2\omega^2x_0^2\sin^2\omega t}=\sqrt{\frac{\hbar m\omega}{2}}.$$

The uncertainty product is consistent with Heisenberg's principle ($\sigma_x\sigma_p\geq\hbar/2$).

$$\sigma_x\sigma_p=\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}}=\frac{\hbar}{2}$$

Part (d)

Here the solution given in part (a) will be derived. The aim is to solve the Schrödinger equation with a harmonic oscillator potential for the wave function $\Psi(x,t)$ that is initially a gaussian wave packet centered at $x = x_0$.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$
$$\Psi(x,0) = A\exp\left[-\frac{m\omega}{2\hbar}(x-x_0)^2\right]$$

Normalizing the initial wave function yields

$$1 = \int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx = A^2\int_{-\infty}^{\infty}\exp\left[-\frac{m\omega}{\hbar}(x-x_0)^2\right]dx = A^2\sqrt{\frac{\pi\hbar}{m\omega}} \quad \rightarrow \quad A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$

Observe that if $x_0 = 0$, then $\Psi(x,0)$ would be the ground state of the harmonic oscillator, and the solution to the Schrödinger equation would be

$$\Psi(x,t) = \psi_0(x)e^{-iE_0t/\hbar} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)e^{-i\omega t/2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2 - \frac{i\omega}{2}t\right).$$

This prompts the substitution,

$$\Psi(x,t) = Ae^{u(x,t)}.$$

Calculate the derivatives of Ψ.

$$\frac{\partial\Psi}{\partial t} = Ae^u\frac{\partial u}{\partial t}$$
$$\frac{\partial\Psi}{\partial x} = Ae^u\frac{\partial u}{\partial x}$$
$$\frac{\partial^2\Psi}{\partial x^2} = A\left[e^u\left(\frac{\partial u}{\partial x}\right)^2 + e^u\frac{\partial^2 u}{\partial x^2}\right] = Ae^u\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial x^2}\right]$$

Substitute these formulas for Ψ and its derivatives into the PDE.

$$i\hbar Ae^u\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}Ae^u\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial x^2}\right] + \frac{1}{2}m\omega^2x^2Ae^u$$

Cancel Ae^u from both sides.

$$i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial x^2}\right] + \frac{1}{2}m\omega^2x^2, \quad -\infty < x < \infty,\ t > 0$$

The initial condition associated with this nonlinear inhomogeneous PDE for u is

$$\Psi(x,0) = Ae^{u(x,0)} = A\exp\left[-\frac{m\omega}{2\hbar}(x-x_0)^2\right] \quad \Rightarrow \quad u(x,0) = -\frac{m\omega}{2\hbar}(x-x_0)^2.$$

Divide both sides of the PDE by $i\hbar$.

$$\frac{\partial u}{\partial t} = \frac{i\hbar}{2m}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial x^2}\right] - \frac{im\omega^2 x^2}{2\hbar}$$

Integrate both sides partially with respect to time from 0 to t.

$$\begin{aligned}
u(x,t) - u(x,0) &= \int_0^t \left\{\frac{i\hbar}{2m}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial x^2}\right] - \frac{im\omega^2 x^2}{2\hbar}\right\}\bigg|_{t=s} ds \\
&= \frac{i\hbar}{2m}\int_0^t \left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial x^2}\right] ds - \frac{im\omega^2 x^2}{2\hbar}\int_0^t ds \\
&= \frac{i\hbar}{2m}\left[\int_0^t \left(\frac{\partial u}{\partial x}\right)^2 ds + \int_0^t \frac{\partial^2 u}{\partial x^2}\, ds\right] - \frac{im\omega^2 x^2 t}{2\hbar}
\end{aligned}$$

Solve for $u(x,t)$.

$$u(x,t) = u(x,0) - \frac{im\omega^2 x^2 t}{2\hbar} + \frac{i\hbar}{2m}\left[\int_0^t \left(\frac{\partial u}{\partial x}\right)^2 ds + \int_0^t \frac{\partial^2 u}{\partial x^2}\, ds\right]$$

Plug in the initial condition.

$$u(x,t) = -\frac{m\omega}{2\hbar}(x-x_0)^2 - \frac{im\omega^2 x^2 t}{2\hbar} + \frac{i\hbar}{2m}\left[\int_0^t \left(\frac{\partial u}{\partial x}\right)^2 ds + \int_0^t \frac{\partial^2 u}{\partial x^2}\, ds\right] \tag{1}$$

Now apply the Adomian decomposition method: Assume that $u(x,t)$ can be decomposed into an infinite number of components, $u_0(x,t)$, $u_1(x,t)$, $\ldots$, such that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + \cdots.$$

Consequently, the nonlinear term can be expressed as

$$\begin{aligned}
\left(\frac{\partial u}{\partial x}\right)^2 &= \left[\frac{\partial}{\partial x}(u_0 + u_1 + u_2 + u_3 + \cdots)\right]^2 \\
&= \left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} + \cdots\right)^2 \\
&= \underbrace{\left(\frac{\partial u_0}{\partial x}\right)^2}_{A_0(x,t)} + \underbrace{2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x}}_{A_1(x,t)} + \underbrace{2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x} + \left(\frac{\partial u_1}{\partial x}\right)^2}_{A_2(x,t)} + \underbrace{2\frac{\partial u_0}{\partial x}\frac{\partial u_3}{\partial x} + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}}_{A_3(x,t)} + \cdots \\
&= A_0 + A_1 + A_2 + A_3 + \cdots,
\end{aligned}$$

where A_0, A_1, $\ldots$, A_n are the Adomian polynomials for $(\partial u/\partial x)^2$, chosen to be the sum of terms whose indices sum to n.

With this decomposition for u, equation (1) becomes

$$u_0 + u_1 + \cdots = -\frac{m\omega}{2\hbar}(x - x_0)^2 - \frac{im\omega^2 x^2 t}{2\hbar} + \frac{i\hbar}{2m}\left[\int_0^t (A_0 + A_1 + \cdots)\,ds + \int_0^t \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} + \cdots\right) ds\right].$$

If we set

$$u_0(x,t) = -\frac{m\omega}{2\hbar}(x - x_0)^2 - \frac{im\omega^2 x^2 t}{2\hbar},$$

then the rest of the components can be determined in a recursive manner.

$$\begin{aligned}
u_1(x,t) &= \frac{i\hbar}{2m}\left[\int_0^t A_0(x,s)\,ds + \int_0^t \frac{\partial^2 u_0}{\partial x^2}(x,s)\,ds\right] \\
u_2(x,t) &= \frac{i\hbar}{2m}\left[\int_0^t A_1(x,s)\,ds + \int_0^t \frac{\partial^2 u_1}{\partial x^2}(x,s)\,ds\right] \\
u_3(x,t) &= \frac{i\hbar}{2m}\left[\int_0^t A_2(x,s)\,ds + \int_0^t \frac{\partial^2 u_2}{\partial x^2}(x,s)\,ds\right] \\
&\vdots \\
u_{n+1} &= \frac{i\hbar}{2m}\left[\int_0^t A_n(x,s)\,ds + \int_0^t \frac{\partial^2 u_n}{\partial x^2}(x,s)\,ds\right], \quad n \geq 0
\end{aligned}$$

Start by calculating the first component.

$$\begin{aligned}
u_1(x,t) &= \frac{i\hbar}{2m}\left[\int_0^t A_0(x,s)\,ds + \int_0^t \frac{\partial^2 u_0}{\partial x^2}(x,s)\,ds\right] \\
&= \frac{i\hbar}{2m}\left[\int_0^t \frac{\partial u_0}{\partial x}(x,s)\frac{\partial u_0}{\partial x}(x,s)\,ds + \int_0^t \frac{\partial^2 u_0}{\partial x^2}(x,s)\,ds\right] \\
&= \frac{i\hbar}{2m}\left\{\int_0^t \left[-\frac{m\omega(x - x_0)}{\hbar} - \frac{im\omega^2 xs}{\hbar}\right]^2 ds + \int_0^t \left(-\frac{m\omega}{\hbar} - \frac{im\omega^2 s}{\hbar}\right) ds\right\} \\
&= -\frac{i\omega t}{2} + \frac{\omega^2 t^2}{4} + \frac{im\omega^2 x^2 t}{2\hbar} - \frac{im\omega^2 x_0 x t}{\hbar} + \frac{im\omega^2 x_0^2 t}{2\hbar} - \frac{m\omega^3 x^2 t^2}{2\hbar} + \frac{m\omega^3 x_0 x t^2}{2\hbar} - \frac{im\omega^4 x^2 t^3}{6\hbar}
\end{aligned}$$

Calculate the second component.

$$\begin{aligned}
u_2(x,t) &= \frac{i\hbar}{2m}\left[\int_0^t A_1(x,s)\,ds + \int_0^t \frac{\partial^2 u_1}{\partial x^2}(x,s)\,ds\right] \\
&= \frac{i\hbar}{2m}\left[2\int_0^t \frac{\partial u_0}{\partial x}(x,s)\frac{\partial u_1}{\partial x}(x,s)\,ds + \int_0^t \frac{\partial^2 u_1}{\partial x^2}(x,s)\,ds\right] \\
&= -\frac{\omega^2 t^2}{4} - \frac{i\omega^3 t^3}{6} + \frac{\omega^4 t^4}{24} + \frac{m\omega^3 x^2 t^2}{2\hbar} - \frac{m\omega^3 x_0 x t^2}{\hbar} + \frac{m\omega^3 x_0^2 t^2}{2\hbar} + \frac{2im\omega^4 x^2 t^3}{3\hbar} \\
&\quad - \frac{5im\omega^4 x_0 x t^3}{6\hbar} + \frac{im\omega^4 x_0^2 t^3}{6\hbar} - \frac{m\omega^5 x^2 t^4}{3\hbar} + \frac{5m\omega^5 x_0 x t^4}{24\hbar} - \frac{im\omega^6 x^2 t^5}{15\hbar}
\end{aligned}$$

Calculate the third component.

$$\begin{aligned}
u_3(x,t) &= \frac{i\hbar}{2m}\left[\int_0^t A_2(x,s)\,ds + \int_0^t \frac{\partial^2 u_2}{\partial x^2}(x,s)\,ds\right] \\
&= \frac{i\hbar}{2m}\left\{\int_0^t \left[2\frac{\partial u_0}{\partial x}(x,s)\frac{\partial u_2}{\partial x}(x,s) + \frac{\partial u_1}{\partial x}(x,s)\frac{\partial u_1}{\partial x}(x,s)\right] ds + \int_0^t \frac{\partial^2 u_2}{\partial x^2}(x,s)\,ds\right\} \\
&= \frac{i\omega^3t^3}{6} - \frac{\omega^4t^4}{6} - \frac{i\omega^5t^5}{15} + \frac{\omega^6t^6}{90} - \frac{im\omega^4x^2t^3}{2\hbar} + \frac{im\omega^4x_0xt^3}{\hbar} - \frac{im\omega^4x_0^2t^3}{2\hbar} + \frac{5m\omega^5x^2t^4}{6\hbar} \\
&\quad - \frac{7m\omega^5x_0xt^4}{6\hbar} + \frac{m\omega^5x_0^2t^4}{3\hbar} + \frac{17im\omega^6x^2t^5}{30\hbar} - \frac{61im\omega^6x_0xt^5}{120\hbar} + \frac{im\omega^6x_0^2t^5}{15\hbar} \\
&\quad - \frac{17m\omega^7x^2t^6}{90\hbar} + \frac{61m\omega^7x_0xt^6}{720\hbar} - \frac{17im\omega^8x^2t^7}{630\hbar}
\end{aligned}$$

Add all the components together to get u.

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \\
&= -\frac{i\omega t}{2} - \frac{\omega^4t^4}{8} - \frac{i\omega^5t^5}{15} + \frac{\omega^6t^6}{90} - \frac{m\omega x^2}{2\hbar} + \frac{m\omega x_0x}{\hbar} - \frac{m\omega x_0^2}{2\hbar} - \frac{im\omega^2x_0xt}{\hbar} \\
&\quad + \frac{im\omega^2x_0^2t}{2\hbar} - \frac{m\omega^3x_0xt^2}{2\hbar} + \frac{m\omega^3x_0^2t^2}{2\hbar} + \frac{im\omega^4x_0xt^3}{6\hbar} - \frac{im\omega^4x_0^2t^3}{3\hbar} \\
&\quad + \frac{m\omega^5x^2t^4}{2\hbar} - \frac{23m\omega^5x_0xt^4}{24\hbar} + \frac{m\omega^5x_0^2t^4}{3\hbar} + \frac{im\omega^6x^2t^5}{2\hbar} - \frac{61im\omega^6x_0xt^5}{120\hbar} \\
&\quad + \frac{im\omega^6x_0^2t^5}{15\hbar} - \frac{17m\omega^7x^2t^6}{90\hbar} + \frac{61m\omega^7x_0xt^6}{720\hbar} - \frac{17im\omega^8x^2t^7}{630\hbar} + \cdots \\
&= -\frac{m\omega}{2\hbar}\left(\frac{i\hbar t}{m} + x^2 - 2xx_0 + x_0^2 + 2i\omega t x x_0 - i\omega t x_0^2 + \omega^2t^2xx_0 - \omega^2t^2x_0^2 - \frac{i\omega^3t^3xx_0}{3} + \frac{2i\omega^3t^3x_0^2}{3}\right. \\
&\quad - \omega^4t^4x^2 + \frac{23\omega^4t^4xx_0}{12} - \frac{2\omega^4t^4x_0^2}{3} - i\omega^5t^5x^2 + \frac{61i\omega^5t^5xx_0}{60} - \frac{2i\omega^5t^5x_0^2}{15} \\
&\quad \left. + \frac{17\omega^6t^6x^2}{45} - \frac{61\omega^6t^6xx_0}{360} + \frac{17i\omega^7t^7x^2}{315} + \frac{\hbar\omega^3t^4}{4m} + \frac{2i\hbar\omega^4t^5}{15m} - \frac{\hbar\omega^5t^6}{45m} + \cdots\right)
\end{aligned}$$

Because only components up to u_3 were calculated, u is accurate only up to the order of t^3.

$$\begin{aligned}
u(x,t) &= -\frac{m\omega}{2\hbar}\left[\frac{i\hbar t}{m} + x^2 - 2xx_0\left(1 - i\omega t - \frac{\omega^2t^2}{2} + \frac{i\omega^3t^3}{6} + \cdots\right) + \frac{x_0^2}{2}\left(2 - 2i\omega t - 2\omega^2t^2 + \frac{4i\omega^3t^3}{3} + \cdots\right)\right] \\
&= -\frac{m\omega}{2\hbar}\left\{\frac{i\hbar t}{m} + x^2 - 2xx_0\left[\frac{(-i\omega t)^0}{0!} + \frac{(-i\omega t)^1}{1!} + \frac{(-i\omega t)^2}{2!} + \frac{(-i\omega t)^3}{3!} + \cdots\right]\right. \\
&\quad \left. + \frac{x_0^2}{2}\left[1 + \frac{(-2i\omega t)^0}{0!} + \frac{(-2i\omega t)^1}{1!} + \frac{(-2i\omega t)^2}{2!} + \frac{(-2i\omega t)^3}{3!} + \cdots\right]\right\} \\
&= -\frac{m\omega}{2\hbar}\left[\frac{i\hbar t}{m} + x^2 - 2xx_0e^{-i\omega t} + \frac{x_0^2}{2}(1 + e^{-2i\omega t})\right]
\end{aligned}$$

Therefore, since $\Psi(x,t) = Ae^{u(x,t)}$,

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{x_0^2}{2}\left(1 + e^{-2i\omega t}\right) + \frac{i\hbar t}{m} - 2x_0xe^{-i\omega t}\right)\right].$$