Exercise 1.2.8

If $u(x,t)$ is known, give an expression for the total thermal energy contained in a rod ($0 < x < L$).

Solution

The thermal energy in the rod is equal to the mass m times specific heat c times temperature u. For a nonuniform rod, the total thermal energy is obtained by integrating over the rod's mass.

$$\text{Total Thermal Energy} = \int_{\text{rod}} cu\,dm$$

The specific heat is assumed to vary as a function of x, and $u = u(x,t)$. The mass is density times volume, so the differential is $dm = \rho(x)\,dV$. The integral changes into one over the rod's volume.

$$\text{Total Thermal Energy} = \int_{\text{rod}} \rho(x)c(x)u(x,t)\,dV$$

Assuming that the cross-sectional area of the rod also varies as a function of x, the differential of volume is $dV = A(x)\,dx$. The integral changes into one over the rod's length.

$$\text{Total Thermal Energy} = \int_{\text{rod}} \rho(x)c(x)u(x,t)A(x)\,dx$$

As the rod goes from $0 < x < L$, the total energy is therefore

$$\text{Total Thermal Energy} = \int_0^L \rho(x)c(x)u(x,t)A(x)\,dx.$$