Exercise 1.2.7

Consider conservation of thermal energy (1.2.4) for any segment of a one-dimensional rod $a \leq x \leq b$. By using the fundamental theorem of calculus,

$$\frac{\partial}{\partial b}\int_a^b f(x)\,dx = f(b),$$

derive the heat equation (1.2.9).

Solution

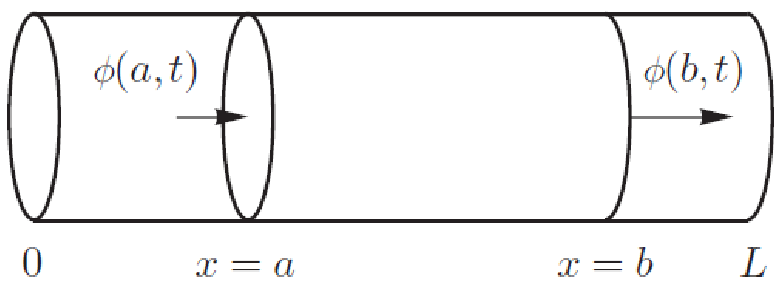

Figure 1: This is a schematic of the one-dimensional rod in question.

Equation (1.2.4) in the text is the law of conservation of thermal energy applied to a one-dimensional rod with a heat source $Q(x,t)$.

$$\frac{d}{dt}\int_a^b e\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q\,dx \tag{1.2.4}$$

Bring the derivative inside the integral on the left side and factor a minus sign from the terms containing ϕ.

$$\int_a^b \frac{\partial e}{\partial t}\,dx = -[\phi(b,t) - \phi(a,t)] + \int_a^b Q\,dx$$

The term in square brackets can be written as an integral by another part of the fundamental theorem of calculus.

$$\int_a^b \frac{\partial e}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b Q\,dx$$

Differentiate both sides with respect to b.

$$\frac{\partial}{\partial b}\int_a^b \frac{\partial e}{\partial t}\,dx = \frac{\partial}{\partial b}\left(-\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b Q\,dx\right)$$

Split up the derivative on the right side into two.

$$\frac{\partial}{\partial b}\int_a^b \frac{\partial e}{\partial t}\,dx = -\frac{\partial}{\partial b}\int_a^b \frac{\partial \phi}{\partial x}\,dx + \frac{\partial}{\partial b}\int_a^b Q\,dx$$

Apply the part of the fundamental theorem of calculus in the problem statement three times here.

$$\frac{\partial e}{\partial t}(b,t) = -\frac{\partial \phi}{\partial b}(b,t) + Q(b,t) \tag{1}$$

The thermal energy in the rod is equal to the mass m times specific heat c times temperature $u(x,t)$. $e(x,t)$ represents the thermal energy density (thermal energy per unit volume). If we integrate it over the volume between $x=a$ and $x=b$, then we will get the total thermal energy in that region.

$$\int_{\text{rod}} e(x,t)\,dV = mcu(x,t)$$

The right side can be written as a volume integral as well because mass is density times volume. For a nonuniform one-dimensional rod, the specific heat and density vary as a function of x, $c=c(x)$ and $\rho=\rho(x)$, respectively.

$$\int_{\text{rod}} e(x,t)\,dV = \int_{\text{rod}} \rho(x)c(x)u(x,t)\,dV$$

Assuming that the cross-sectional area A of the rod is constant, the differential of volume is $dV = A\,dx$.

$$\int_a^b e(x,t)A\,dx = \int_a^b \rho(x)c(x)u(x,t)A\,dx$$

Since the two integrals are equal over the same interval of x, the integrands must be equal.

$$e(x,t)A = \rho(x)c(x)u(x,t)A$$

Divide both sides by A to get the formula for $e(x,t)$.

$$e(x,t) = \rho(x)c(x)u(x,t)$$

Substitute it into equation (1).

$$\frac{\partial}{\partial t}[\rho(b)c(b)u(b,t)] = -\frac{\partial \phi}{\partial b}(b,t) + Q(b,t)$$

$\rho(b)$ and $c(b)$ are constant in time and can be pulled in front of the time derivative.

$$\rho(b)c(b)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial b}(b,t) + Q(b,t)$$

According to Fourier's law of heat conduction, the heat flux is proportional to the temperature gradient.

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x},$$

where $K_0(x)$ is a proportionality constant known as the thermal conductivity. It varies as a function of x because the rod is nonuniform. As a result, the energy balance becomes an equation solely for the temperature.

$$\rho(b)c(b)\frac{\partial u}{\partial t} = -\frac{\partial}{\partial b}\left[-K_0(b)\frac{\partial u}{\partial b}\right] + Q(b,t)$$

Bring the minus sign out of the derivative and change the dummy variable for position from b to x. This gives us equation (1.2.9) in the text.

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q \tag{1.2.9}$$