Exercise 1.2.9

Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x), \tag{1.2.15}$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare (1.2.15) with the equation for a one-dimensional rod whose lateral surfaces are insulated, but with heat sources.

(d) Specialize (1.2.15) to a rod of circular cross section with constant thermal properties and 0° outside temperature.

(e) Consider the assumptions in part (d). Suppose that the temperature in the rod is uniform [i.e., $u(x,t) = u(t)$]. Determine $u(t)$ if initially $u(0) = u_0$.

Solution

Part (a)

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters the left side of a rod at $x = a$, then that same amount must exit the right side of it at $x = b$ for the temperature to remain the same. If more (less) thermal energy enters at $x = a$ than exits at $x = b$, then the amount of thermal energy in the rod will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of thermal energy in} - \text{rate of thermal energy out} = \text{rate of energy accumulation}$$

The loss of thermal energy from the lateral sides will be included on the left side as one of the terms for "rate of thermal energy out." Since $w(x,t)$ is the rate of heat loss per unit surface area, it has to be integrated over the rod's surface area to get the total. The heat flux is defined to be the rate that thermal energy flows through the rod per unit area, and we denote it by $\phi = \phi(x,t)$. If we let U represent the amount of thermal energy of the rod, then the energy balance over it is

$$A(a)\phi(a,t) - A(b)\phi(b,t) - \int_{\text{rod}} w(x,t)\,dA = \left.\frac{dU}{dt}\right|_{\text{rod}}.$$

The surface area differential can be written in terms of the perimeter $P(x)$ as $dA = P(x)\,dx$. It is assumed here that the rod has a perimeter that varies with x and that is small enough such that thermal energy essentially flows in the x-direction.

$$A(a)\phi(a,t) - A(b)\phi(b,t) - \int_a^b w(x,t)P(x)\,dx = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

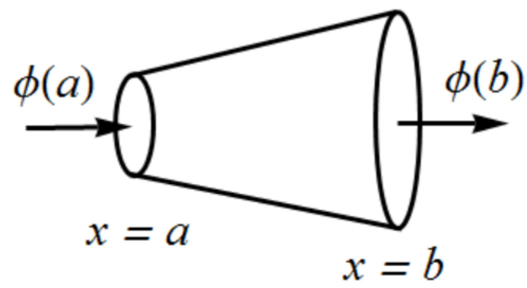

Figure 1: This is a schematic of the rod in question. It has variable cross-sectional area and lateral perimeter and physical properties. The heat flow into the left side at $x = a$ is the cross-sectional area there $A(a)$ times $\phi(a, t)$, and the heat flow out of the right side at $x = b$ is the cross-sectional area there $A(b)$ times $\phi(b, t)$.

Factor a minus sign from the two terms containing ϕ.

$$-[A(b)\phi(b,t) - A(a)\phi(a,t)] - \int_a^b w(x,t)P(x)\,dx = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

By the fundamental theorem of calculus, the term in square brackets can be written as an integral.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\phi(x,t)]\,dx - \int_a^b w(x,t)P(x)\,dx = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

Combine the two integrals on the left side.

$$\int_a^b \left\{-\frac{\partial}{\partial x}[A(x)\phi(x,t)] - w(x,t)P(x)\right\} dx = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

The thermal energy in the rod U is equal to the mass m times specific heat c times temperature $u(x,t)$. As the rod is nonuniform, the total thermal energy is obtained by integrating over the rod's mass.

$$\int_a^b \left\{-\frac{\partial}{\partial x}[A(x)\phi(x,t)] - w(x,t)P(x)\right\} dx = \frac{d}{dt}\int_{\text{rod}} c(x)u(x,t)\,dm$$

The mass is density times volume, so the differential is $dm = \rho(x)\,dV$. The volume differential itself can be written in terms of the cross-sectional area $A(x)$ as $dV = A(x)\,dx$.

$$\int_a^b \left\{-\frac{\partial}{\partial x}[A(x)\phi(x,t)] - w(x,t)P(x)\right\} dx = \frac{d}{dt}\int_a^b \rho(x)c(x)u(x,t)A(x)\,dx$$

Bring the derivative inside the integral on the right side.

$$\int_a^b \left\{-\frac{\partial}{\partial x}[A(x)\phi(x,t)] - w(x,t)P(x)\right\} dx = \int_a^b \rho(x)c(x)\frac{\partial u}{\partial t}A(x)\,dx$$

The two integrals are equal over the same interval of x, so the integrands must be equal.

$$-\frac{\partial}{\partial x}[A(x)\phi(x,t)] - w(x,t)P(x) = \rho(x)c(x)\frac{\partial u}{\partial t}A(x)$$

According to Fourier's law of heat conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0(x)\frac{\partial u}{\partial x},$$

where $K_0(x)$ is a proportionality constant known as the thermal conductivity. It varies as a function of x because the rod is nonuniform. As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial x}\left[-A(x)K_0(x)\frac{\partial u}{\partial x}\right] - w(x,t)P(x) = \rho(x)c(x)\frac{\partial u}{\partial t}A(x)$$

Therefore, after dividing both sides by $A(x)$, the partial differential equation for the temperature in a nonuniform rod with cross-sectional area and lateral perimeter that vary with x is

$$\rho(x)c(x)\frac{\partial u}{\partial t} = \frac{1}{A(x)}\frac{\partial}{\partial x}\left[A(x)K_0(x)\frac{\partial u}{\partial x}\right] - \frac{P(x)}{A(x)}w(x,t).$$

Part (b)

Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. This is Newton's law of cooling.

$$w(x,t) \propto u(x,t) - \gamma(x,t)$$

This proportionality can be changed to an equation by introducing a proportionality function $h(x)$. It is not a constant because the rod is nonuniform.

$$w(x,t) = h(x)[u(x,t) - \gamma(x,t)]$$

Substitute this formula for $w(x,t)$ into the PDE derived in part (a).

$$\rho(x)c(x)\frac{\partial u}{\partial t} = \frac{1}{A(x)}\frac{\partial}{\partial x}\left[A(x)K_0(x)\frac{\partial u}{\partial x}\right] - \frac{P(x)}{A(x)}h(x)[u(x,t) - \gamma(x,t)]$$

Assuming now that the rod has constant cross-sectional area A and constant lateral perimeter P and constant mass density ρ and constant specific heat c and constant thermal conductivity K_0, we therefore obtain

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x).$$

Part (c)

Comparing this to the heat equation for an insulated rod with a heat source,

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q,$$

we see that a rod without insulation satisfies the same equation. Rather than a heat source, the lack of insulation is effectively a heat sink because energy is lost to the environment in proportion to the temperature difference $u(x,t) - \gamma(x,t)$, the ease with which energy transfers to the environment $h(x)$, and the surface-area-to-volume ratio P/A (multiply the numerator and denominator by the rod length).

Part (d)

Making the additional assumptions that $\gamma(x,t) = 0$ and $A = \pi r^2$ and $P = 2\pi r$ and $h(x)$ is constant, the PDE at the end of part (b) simplifies to

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} - \frac{2h}{r} u(x,t).$$

Part (e)

Making the additional assumption that u is the same throughout the rod, that is, $u = u(t)$, the second derivative with respect to x vanishes.

$$c\rho \frac{\partial u}{\partial t} = \underbrace{K_0 \frac{\partial^2 u}{\partial x^2}}_{=0} - \frac{2h}{r} u(t)$$

Consequently, an ordinary differential equation emerges.

$$c\rho \frac{du}{dt} = -\frac{2h}{r} u(t)$$

Divide both sides by $c\rho u(t)$.

$$\frac{1}{u}\frac{du}{dt} = -\frac{2h}{c\rho r}$$

The left side can be written as the derivative of $\ln u$.

$$\frac{d}{dt}(\ln u) = -\frac{2h}{c\rho r}$$

Integrate both sides with respect to t.

$$\ln u = -\frac{2h}{c\rho r} t + C$$

Exponentiate both sides.

$$\begin{aligned} u(t) &= \exp\left(-\frac{2h}{c\rho r} t + C\right) \\ &= \exp(C) \exp\left(-\frac{2h}{c\rho r} t\right) \end{aligned}$$

Introduce a new constant of integration B.

$$= B \exp\left(-\frac{2h}{c\rho r} t\right)$$

Apply the prescribed initial condition $u(0) = u_0$ to determine B.

$$u(0) = B = u_0$$

Therefore, the temperature of a uniform cylindrical rod falls exponentially from u_0 to 0°.

$$u(t) = u_0 \exp\left(-\frac{2h}{c\rho r} t\right)$$