Exercise 1.3.1

Consider a one-dimensional rod, $0 \leq x \leq L$. Assume that the heat energy flowing out of the rod at $x = L$ is proportional to the temperature difference between the end temperature of the bar and the known external temperature. Derive (1.3.5); briefly, physically explain why $H > 0$;

[I think a better way to write the last independent clause is "explain briefly why $H > 0$ physically."]

Solution

From the assumption a proportionality can be made. The heat energy flowing out of $x = L$ is the heat flux ϕ there times the cross-sectional area A. Let the ambient temperature be denoted by $u_B = u_B(t)$.

$$A\phi(L,t) \propto u(L,t) - u_B(t)$$

To change this to an equation, we introduce a proportionality constant h.

$$A\phi(L,t) = h[u(L,t) - u_B(t)]$$

Divide both sides by A and let $H = h/A$. This is known as the heat transfer coefficient—it's a measure of how easy it is for energy to be exchanged between the rod and the environment.

$$\phi(L,t) = H[u(L,t) - u_B(t)]$$

According to Fourier's law of heat conduction, the heat flux is proportional to the temperature gradient.

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x},$$

where $K_0(x)$ is a proportionality constant known as the thermal conductivity. It varies as a function of x if we assume the rod is nonuniform. Substitute this formula into the left side for $\phi(L,t)$.

$$-K_0(L)\frac{\partial u}{\partial x}(L,t) = H[u(L,t) - u_B(t)] \tag{1.3.5}$$

$H > 0$ because when the rod is hotter than the environment $u(L,t) - u_B(t) > 0$, heat will flow towards the right, which is what the positive x-direction is in. Conversely, if the rod is cooler than the ambient temperature $u(L,t) - u_B(t) < 0$, then heat will flow to the left, which is in the negative x-direction.