Exercise 1.4.10

Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, $\frac{\partial u}{\partial x}(L,t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

Solution

The governing equation for the rod's temperature u is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4.$$

Comparing this to the general form of the heat equation, we see that the mass density ρ and specific heat c are equal to 1 and that the heat source is $Q = 4$. The thermal energy density e is $\rho c u = u$, so the left side can be written in terms of e.

$$\frac{\partial e}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$$

To obtain the total thermal energy in the rod, integrate both sides over the rod's volume V.

$$\int_V \frac{\partial e}{\partial t}\,dV = \int_V \left(\frac{\partial^2 u}{\partial x^2} + 4\right) dV$$

Bring the time derivative in front of the volume integral on the left.

$$\frac{d}{dt}\int_V e\,dV = \int_V \left(\frac{\partial^2 u}{\partial x^2} + 4\right) dV$$

The volume integral on the left represents the total thermal energy in the rod, and that's what we intend to solve for. The rod has a constant cross-sectional area A, so the volume differential is $dV = A\,dx$. The volume integral on the right side will be replaced by one over the rod's length.

$$\begin{aligned}
\frac{d}{dt}\int_V e\,dV &= \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + 4\right) A\,dx \\
&= A\left(\int_0^L \frac{\partial^2 u}{\partial x^2}\,dx + 4\int_0^L dx\right) \\
&= A\left(\frac{\partial u}{\partial x}\bigg|_0^L + 4L\right) \\
&= A\Bigg[\underbrace{\frac{\partial u}{\partial x}(L,t)}_{=6} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=5} + 4L\Bigg] \\
&= A(1+4L)
\end{aligned}$$

Integrate both sides with respect to t.

$$\int_V e\,dV = A(1+4L)t + U_0$$

The constant of integration U_0 is the initial thermal energy in the rod. In order to determine it, we will make use of the initial condition $u(x,0) = f(x)$. Change e back in terms of u and write $dV = A\,dx$.

$$\int_0^L u(x,t)A\,dx = A(1+4L)t + U_0$$

Bring A in front of the integral and set $t = 0$ in the equation.

$$A \int_0^L u(x, 0)\, dx = U_0$$

Use the initial condition.

$$A \int_0^L f(x)\, dx = U_0$$

Therefore, the thermal energy in the rod as a function of time is

$$\int_V e\, dV = A(1 + 4L)t + A \int_0^L f(x)\, dx.$$