Exercise 1.4.3

Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material $(c\rho = 1,\ K_0 = 1)$ with a constant source $(Q = 1)$, whereas for the other $1 < x < 2$, there are no sources $(Q = 0,\ c\rho = 2,\ K_0 = 2)$ (see Exercise 1.3.2) with $u(0) = 0$ and $u(2) = 0$.

Solution

The governing equation for the temperature in a one-dimensional rod with constant physical properties and a heat source Q is the heat equation.

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q$$

The heat equation applies to each segment of the rod.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, & 0 < x < 1 \\ 2\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, & 1 < x < 2 \end{cases}$$

The heat flux ϕ is defined as the rate of thermal energy flowing per unit area. According to Fourier's law of conduction, it is proportional to the temperature gradient.

$$\phi = -K_0(x)\frac{\partial u}{\partial x}$$

If the two materials of the rod are in perfect thermal contact at $x = 1$, then the temperature is not only continuous there,

$$\lim_{x\to 1^-} u(x,t) = \lim_{x\to 1^+} u(x,t), \tag{1}$$

but also the rate of heat flowing from the left must be equal to the rate of heat flowing to the right.

$$\lim_{x\to 1^-} A\phi(x,t) = \lim_{x\to 1^+} A\phi(x,t)$$

Using Fourier's law for the flux, this boundary condition becomes

$$\lim_{x\to 1^-} -AK_0(x)\frac{\partial u}{\partial x} = \lim_{x\to 1^+} -AK_0(x)\frac{\partial u}{\partial x}.$$

The rod has constant cross-sectional area A in both materials but different thermal conductivities. Dividing both sides by $-A$, the second boundary condition is thus

$$\lim_{x\to 1^-} (1)\frac{\partial u}{\partial x} = \lim_{x\to 1^+} (2)\frac{\partial u}{\partial x}. \tag{2}$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$\begin{cases} 0 = \dfrac{d^2 u}{dx^2} + 1, & 0 < x < 1 \\ 0 = 2\dfrac{d^2 u}{dx^2}, & 1 < x < 2 \end{cases}$$

The general solution to both ODEs can be obtained by integrating both sides with respect to x twice. After the first integration, we get

$$\begin{cases} \dfrac{du}{dx} = -x + C_1, & 0 < x < 1 \\ \dfrac{du}{dx} = C_2, & 1 < x < 2 \end{cases}.$$

Apply equation (2) here to determine one of the constants.

$$(1)(-1 + C_1) = (2)C_2$$

As a result, $C_2 = (-1 + C_1)/2$.

$$\begin{cases} \dfrac{du}{dx} = -x + C_1, & 0 < x < 1 \\ \dfrac{du}{dx} = \dfrac{-1 + C_1}{2}, & 1 < x < 2 \end{cases}$$

Integrate both sides with respect to x once more.

$$\begin{cases} u(x) = -\dfrac{x^2}{2} + C_1 x + C_3, & 0 < x < 1 \\ u(x) = \dfrac{-1 + C_1}{2} x + C_4, & 1 < x < 2 \end{cases}$$

Apply the boundary conditions, $u(0) = 0$ and $u(2) = 0$, here to determine two more constants.

$$\begin{aligned} u(0) &= C_3 = 0 \\ u(2) &= \frac{-1 + C_1}{2} \cdot 2 + C_4 = 0 \end{aligned}$$

Solving the second equation for C_4 gives $C_4 = 1 - C_1$.

$$\begin{cases} u(x) = -\dfrac{x^2}{2} + C_1 x, & 0 < x < 1 \\ u(x) = \dfrac{-1 + C_1}{2} x + (1 - C_1), & 1 < x < 2 \end{cases}$$

Use equation (1) to determine the last constant.

$$-\frac{1}{2} + C_1 = \frac{-1 + C_1}{2} + (1 - C_1) \quad \rightarrow \quad C_1 = \frac{2}{3}$$

Plugging in $C_1 = 2/3$ gives the solution for $u(x)$. Therefore,

$$u(x) = \begin{cases} -\dfrac{x^2}{2} + \dfrac{2}{3} x, & 0 < x < 1 \\ -\dfrac{1}{6} x + \dfrac{1}{3}, & 1 < x < 2 \end{cases}.$$