Exercise 1.4.5

Consider a one-dimensional rod $0 \leq x \leq L$ of known length and known constant thermal properties without sources. Suppose that the temperature is an *unknown* constant T at $x = L$. Determine T if we know (in the steady state) both the temperature and the heat flow at $x = 0$.

Solution

The governing equation for the temperature in a one-dimensional rod with constant physical properties and no heat source is the heat equation.

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}$$

The heat flux ϕ is defined as the rate of thermal energy flowing per unit area. According to Fourier's law of conduction, it is proportional to the temperature gradient.

$$\phi = -K_0 \frac{\partial u}{\partial x}$$

The heat flow at $x = 0$ is then

$$A\phi(0,t) = -AK_0 \frac{\partial u}{\partial x}(0,t) = F \quad \rightarrow \quad \frac{\partial u}{\partial x}(0,t) = -\frac{F}{AK_0}, \tag{1}$$

where F is a known constant. The temperature at $x = 0$ is

$$u(0,t) = T_0, \tag{2}$$

where T_0 is a known constant. Equations (1) and (2) are the boundary conditions for the PDE. In the steady state the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice. After the first integration, we get

$$\frac{du}{dx} = C_1.$$

Apply equation (1) to determine C_1.

$$\frac{du}{dx}(0) = C_1 = -\frac{F}{AK_0}$$

So we have

$$\frac{du}{dx} = -\frac{F}{AK_0}.$$

Integrate both sides with respect to x once more.

$$u(x) = -\frac{F}{AK_0} x + C_2$$

Apply equation (2) to determine C_2.

$$u(0) = C_2 = T_0$$

As a result, the steady-state temperature is

$$u(x) = -\frac{F}{AK_0}x + T_0.$$

The unknown temperature at $x = L$ can now be found.

$$u(L) = -\frac{F}{AK_0}L + T_0$$

Therefore,

$$T = T_0 - \frac{FL}{AK_0},$$

where T_0 is the temperature at $x = 0$ and F is the heat flow at $x = 0$.