Exercise 1.4.11

Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = \beta$, $\frac{\partial u}{\partial x}(L,t) = 7$.

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t\to\infty} u(x,t)$.

Part (a)

The governing equation for the rod's temperature u is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x.$$

Comparing this to the general form of the heat equation, we see that the mass density ρ and specific heat c are equal to 1 and that the heat source is $Q = x$. The thermal energy density e is $\rho c u = u$, so the left side can be written in terms of e.

$$\frac{\partial e}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$$

To obtain the total thermal energy in the rod, integrate both sides over the rod's volume V.

$$\int_V \frac{\partial e}{\partial t}\, dV = \int_V \left(\frac{\partial^2 u}{\partial x^2} + x\right) dV$$

Bring the time derivative in front of the volume integral on the left.

$$\frac{d}{dt}\int_V e\, dV = \int_V \left(\frac{\partial^2 u}{\partial x^2} + x\right) dV$$

The volume integral on the left represents the total thermal energy in the rod, and that's what we intend to solve for. The rod has a constant cross-sectional area A, so the volume differential is $dV = A\, dx$. The volume integral on the right side will be replaced by one over the rod's length.

$$\begin{aligned}
\frac{d}{dt}\int_V e\, dV &= \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x\right) A\, dx \\
&= A\left(\int_0^L \frac{\partial^2 u}{\partial x^2}\, dx + \int_0^L x\, dx\right) \\
&= A\left(\frac{\partial u}{\partial x}\bigg|_0^L + \frac{L^2}{2}\right) \\
&= A\left[\underbrace{\frac{\partial u}{\partial x}(L,t)}_{=\,7} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=\,\beta} + \frac{L^2}{2}\right] \\
&= A\left(7 - \beta + \frac{L^2}{2}\right)
\end{aligned}$$

Integrate both sides with respect to t.

$$\int_V e\, dV = A\left(7 - \beta + \frac{L^2}{2}\right)t + U_0$$

The constant of integration U_0 is the initial thermal energy in the rod. In order to determine it, we will make use of the initial condition $u(x,0) = f(x)$. Change e back in terms of u and write $dV = A\,dx$.

$$\int_0^L u(x,t)A\,dx = A\left(7 - \beta + \frac{L^2}{2}\right)t + U_0$$

Bring A in front of the integral and set $t = 0$ in the equation.

$$A\int_0^L u(x,0)\,dx = U_0$$

Use the initial condition.

$$A\int_0^L f(x)\,dx = U_0$$

Therefore, the thermal energy in the rod as a function of time is

$$\int_V e\,dV = A\left(7 - \beta + \frac{L^2}{2}\right)t + A\int_0^L f(x)\,dx.$$

Part (b)

Equilibrium can only occur if the thermal energy in the rod is constant. This happens if

$$7 - \beta + \frac{L^2}{2} = 0 \quad \rightarrow \quad \beta = 7 + \frac{L^2}{2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = \frac{d^2u}{dx^2} + x \quad \rightarrow \quad \frac{d^2u}{dx^2} = -x$$

This differential equation can be solved by integrating both sides with respect to x twice. After the first integration, we get

$$\frac{du}{dx} = -\frac{x^2}{2} + C_1.$$

Apply the boundary conditions here to determine C_1.

$$\begin{aligned} \frac{du}{dx}(0) &= C_1 = \beta \\ \frac{du}{dx}(L) &= -\frac{L^2}{2} + C_1 = 7 \quad \rightarrow \quad C_1 = 7 + \frac{L^2}{2} \end{aligned}$$

So then

$$\frac{du}{dx} = -\frac{x^2}{2} + 7 + \frac{L^2}{2}.$$

Integrate both sides with respect to x a second time.

$$u(x) = -\frac{x^3}{6} + \left(7 + \frac{L^2}{2}\right)x + C_2$$

The result from part (a) will be used to determine C_2. If $\beta = 7 + L^2/2$, then it simplifies to

$$\int_V e\,dV = A\int_0^L f(x)\,dx.$$

Change e back to u and dV to $A\,dx$.

$$\int_0^L u(x,t)A\,dx = A\int_0^L f(x)\,dx$$

Divide both sides by A and then set $t=\infty$.

$$\int_0^L u(x,\infty)\,dx = \int_0^L f(x)\,dx$$

Substitute the equilibrium temperature for $u(x,\infty)$.

$$\int_0^L \left[-\frac{x^3}{6} + \left(7+\frac{L^2}{2}\right)x + C_2\right]dx = \int_0^L f(x)\,dx$$

We now have an equation for C_2. Evaluate the integral on the left side.

$$-\frac{L^4}{24} + \left(7+\frac{L^2}{2}\right)\frac{L^2}{2} + C_2 L = \int_0^L f(x)\,dx$$

Simplify the left side.

$$\frac{5L^4}{24} + \frac{7L^2}{2} + C_2 L = \int_0^L f(x)\,dx$$

So we have

$$C_2 = -\frac{5L^3}{24} - \frac{7L}{2} + \frac{1}{L}\int_0^L f(x)\,dx.$$

Therefore, assuming $\beta = 7 + L^2/2$, the equilibrium temperature distribution is

$$u(x) = -\frac{x^3}{6} + \left(7+\frac{L^2}{2}\right)x - \frac{5L^3}{24} - \frac{7L}{2} + \frac{1}{L}\int_0^L f(x)\,dx.$$