Exercise 1.4.2

Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

(a) Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationships should exist between the answers in parts (a) and (b)?

Solution

The governing equation for the temperature in a one-dimensional rod with constant area and thermal properties, ρ, c, and K_0, and a heat source $Q = K_0 x$ is

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + K_0 x.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} + K_0 x \quad \rightarrow \quad \frac{d^2 u}{dx^2} = -x$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = -\frac{x^2}{2} + C_1$$

$$u(x) = -\frac{x^3}{6} + C_1 x + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$u(0) = C_2 = 0$$

$$u(L) = -\frac{L^3}{6} + C_1 L + C_2 = 0$$

Solving the second equation for C_1 gives $C_1 = L^2/6$, so the equilibrium temperature distribution is

$$u(x) = -\frac{x^3}{6} + \frac{L^2}{6} x.$$

Part (a)

Since Q represents the heat energy per unit volume generated in the rod per unit time, the heat energy per unit time is obtained by integrating Q over the rod's volume V.

$$\text{Rate of Heat Energy Generated in Rod} = \int_V Q \, dV$$

For a rod with constant cross-sectional area A, the volume differential is $dV = A\,dx$. The integral becomes one over the rod's length.

$$= \int_0^L QA \, dx$$

Substitute $Q = K_0 x$ and evaluate the integral.

$$\begin{aligned}\text{Rate of Heat Energy Generated in Rod} &= \int_0^L K_0 x A \, dx \\ &= AK_0 \frac{L^2}{2}\end{aligned}$$

Part (b)

The heat flux ϕ is defined as the rate of thermal energy flowing per unit area. According to Fourier's law of conduction, it is proportional to the temperature gradient.

$$\phi = -K_0 \frac{du}{dx}$$

Multiplying ϕ by the cross-sectional area A then gives the rate that heat flows.

$$\begin{aligned}&\text{Rate of heat flow at } x = 0: \quad A\phi(0) = -AK_0 \frac{du}{dx}(0) = -AK_0 \frac{L^2}{6} \\ &\text{Rate of heat flow at } x = L: \quad A\phi(L) = -AK_0 \frac{du}{dx}(L) = -AK_0 \left(-\frac{L^2}{3}\right) = AK_0 \frac{L^2}{3}\end{aligned}$$

The minus sign indicates that heat is flowing to the left at $x = 0$, that is, out of the rod. The fact that the heat flow at $x = L$ is positive means heat is flowing to the right—out of the rod.

Part (c)

For an equilibrium temperature distribution to exist, the amount of thermal energy in the rod must remain constant. This occurs if the rate of thermal energy flowing in the rod is equal to the rate flowing out. The mathematical expression for this idea, an energy balance, is

$$\text{rate of energy in} - \text{rate of energy out} = 0.$$

The heat source will be included as one of the terms for "rate of energy in," and each of the heat flows found in part (b) will be terms in "rate of energy out."

$$AK_0 \frac{L^2}{2} - \left(AK_0 \frac{L^2}{6} + AK_0 \frac{L^2}{3}\right) = 0$$

Therefore, the rate of heat generated in the rod is equal to the rate flowing out of the rod's ends in equilibrium.