Exercise 1.4.4

If both ends of a rod are insulated, derive *from the partial differential equation* that the total thermal energy in the rod is constant.

Solution

The governing equation for the temperature in a one-dimensional rod with constant cross-sectional area, general physical properties, and a heat source $Q(x,t)$ is

$$\rho(x)c(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t).$$

Bring $\rho(x)$ and $c(x)$ inside the time derivative.

$$\frac{\partial}{\partial t}[\rho(x)c(x)u(x,t)] = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t)$$

The product of mass density ρ, specific heat c, and temperature u is the thermal energy density $e(x,t)$.

$$\frac{\partial e}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t)$$

To obtain the total thermal energy in the rod, integrate both sides over the rod's volume V.

$$\int_V \frac{\partial e}{\partial t}\,dV = \int_V \left\{\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t)\right\} dV$$

Bring the time derivative in front of the integral on the left side.

$$\frac{d}{dt}\int_V e\,dV = \int_V \left\{\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t)\right\} dV$$

Note that the integral on the left side represents the total thermal energy in the rod, and the time derivative of that integral represents how fast that quantity is changing. The aim now is to evaluate the integral on the right side. If the rod has constant cross-sectional area A, then the volume differential is $dV = A\,dx$. The volume integral becomes one over the rod's length.

$$\begin{aligned}
\frac{d}{dt}\int_V e\,dV &= \int_0^L \left\{\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t)\right\} A\,dx \\
&= A\left\{\int_0^L \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] dx + \int_0^L Q(x,t)\,dx\right\} \\
&= A\left\{\left[K_0(x)\frac{\partial u}{\partial x}\right]\Bigg|_0^L + \int_0^L Q(x,t)\,dx\right\} \\
&= A\left\{K_0(L)\frac{\partial u}{\partial x}(L,t) - K_0(0)\frac{\partial u}{\partial x}(0,t) + \int_0^L Q(x,t)\,dx\right\}
\end{aligned}$$

The heat flux ϕ is defined as the rate of thermal energy flowing per unit area. According to Fourier's law of conduction, it is proportional to the temperature gradient.

$$\phi = -K_0(x)\frac{\partial u}{\partial x}$$

If the rod's ends at $x = 0$ and $x = L$ are insulated, then the boundary conditions there are

$$A\phi(0,t) = 0 \quad \text{and} \quad A\phi(L,t) = 0,$$

respectively. That is,

$$K_0(0)\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad K_0(L)\frac{\partial u}{\partial x}(L,t) = 0.$$

Consequently, the rate that thermal energy changes in the rod simplifies to

$$\frac{d}{dt}\int_V e\,dV = A\int_0^L Q(x,t)\,dx.$$

Assuming that there are no heat sources in the rod (or alternatively that there are as many heat sinks as there are heat sources in the rod), then $Q(x,t) = 0$ ($\int_0^L Q(x,t)\,dx = 0$).

$$\frac{d}{dt}\int_V e\,dV = 0$$

Integrate both sides with respect to t.

$$\int_V e\,dV = \text{constant}$$

Therefore, the total thermal energy in the rod is constant.