Exercise 1.5.12

Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2 \, dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \partial u/\partial r\big|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$