Exercise 1.5.11

Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad a < r < b,$$

subject to

$$u(r,0) = f(r), \ \frac{\partial u}{\partial r}(a,t) = \beta, \text{ and } \frac{\partial u}{\partial r}(b,t) = 1.$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

Solution

The PDE is the heat equation in cylindrical coordinates. For an equilibrium temperature distribution to exist, the amount of thermal energy in the cylinder must remain constant. This occurs if the rate of thermal energy flowing in the cylinder at $r = a$ is equal to the rate flowing out at $r = b$. The mathematical expression for this idea, an energy balance, is

$$\text{rate of energy in} - \text{rate of energy out} = 0.$$

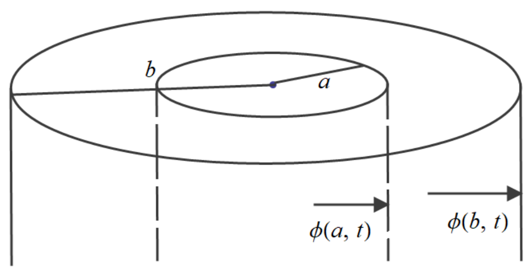

Figure 1: This is a schematic of the cylinder. It has inner radius a, outer radius b, and length L. Thermal energy is assumed to flow radially in the positive r-direction.

The heat flux is defined to be the rate that thermal energy flows through the cylinder per unit area, and we denote it by $\phi = \phi(r,t)$. Multiplying it by the cross-sectional areas at the inner and outer radii then gives the rates in the energy balance.

$$A(a)\phi(a,t) - A(b)\phi(b,t) = 0$$

$$2\pi a L\phi(a,t) - 2\pi b L\phi(b,t) = 0$$

Divide both sides by $2\pi L$ and bring the second term to the right side.

$$a\phi(a,t) = b\phi(b,t)$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi(r,t) = -K_0\frac{\partial u}{\partial r}(r,t),$$

where K_0 is a proportionality constant known as the thermal conductivity. As a result, we have

$$-aK_0\frac{\partial u}{\partial r}(a,t) = -bK_0\frac{\partial u}{\partial r}(b,t).$$

Divide both sides by $-K_0$.

$$a\frac{\partial u}{\partial r}(a,t) = b\frac{\partial u}{\partial r}(b,t)$$

Substitute the boundary conditions, $u_r(a,t) = \beta$ and $u_r(b,t) = 1$, on both sides.

$$a\beta = b$$

Therefore, the condition for an equilibrium temperature distribution is

$$\beta = \frac{b}{a}.$$