Exercise 2.2.1

Show that any linear combination of linear operators is a linear operator.

Solution

Suppose L_1 and L_2 are linear operators. Then, by the definition of linearity,

$$\begin{aligned} L_1(c_1u_1 + c_2u_2) &= c_1L_1(u_1) + c_2L_1(u_2) \\ L_2(c_1u_1 + c_2u_2) &= c_1L_2(u_1) + c_2L_2(u_2), \end{aligned}$$

where c_1 and c_2 are arbitrary constants and u_1 and u_2 are solutions to a linear homogeneous equation. The aim is to show that a linear combination of L_1 and L_2, $c_3L_1 + c_4L_2$, is also linear.

$$(c_3L_1 + c_4L_2)(c_1u_1 + c_2u_2) = c_1(c_3L_1 + c_4L_2)(u_1) + c_2(c_3L_1 + c_4L_2)(u_2).$$

We have

$$\begin{aligned} (c_3L_1 + c_4L_2)(c_1u_1 + c_2u_2) &= c_3L_1(c_1u_1 + c_2u_2) + c_4L_2(c_1u_1 + c_2u_2) \\ &= c_3[c_1L_1(u_1) + c_2L_1(u_2)] + c_4[c_1L_2(u_1) + c_2L_2(u_2)] \\ &= c_3c_1L_1(u_1) + c_3c_2L_1(u_2) + c_4c_1L_2(u_1) + c_4c_2L_2(u_2) \\ &= c_1c_3L_1(u_1) + c_1c_4L_2(u_1) + c_2c_3L_1(u_2) + c_2c_4L_2(u_2) \\ &= c_1[c_3L_1(u_1) + c_4L_2(u_1)] + c_2[c_3L_1(u_2) + c_4L_2(u_2)] \\ &= c_1(c_3L_1 + c_4L_2)(u_1) + c_2(c_3L_1 + c_4L_2)(u_2). \end{aligned}$$

Therefore, any linear combination of linear operators is a linear operator.