Exercise 2.2.2

(a) Show that $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator.

(b) Show that usually $L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is not a linear operator.

Solution

Part (a)

The aim is to show that

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2),$$

where c_1 and c_2 are arbitrary constants and u_1 and u_2 are solutions to a linear homogeneous equation.

$$\begin{aligned}
L(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial(c_1u_1 + c_2u_2)}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[K_0(x)\left(c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x}\right)\right] \\
&= \frac{\partial}{\partial x}\left[c_1K_0(x)\frac{\partial u_1}{\partial x} + c_2K_0(x)\frac{\partial u_2}{\partial x}\right] \\
&= c_1\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_1}{\partial x}\right] + c_2\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_2}{\partial x}\right] \\
&= c_1L(u_1) + c_2L(u_2)
\end{aligned}$$

Therefore, $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator.

Part (b)

The aim is to show that

$$L(c_1u_1 + c_2u_2) \neq c_1L(u_1) + c_2L(u_2),$$

where c_1 and c_2 are arbitrary constants and u_1 and u_2 are solutions to a linear homogeneous equation.

$$\begin{aligned}
L(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial x}\left[K_0(x, c_1u_1 + c_2u_2)\frac{\partial(c_1u_1 + c_2u_2)}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[K_0(x, c_1u_1 + c_2u_2)\left(c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x}\right)\right] \\
&= \frac{\partial}{\partial x}\left[c_1K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_1}{\partial x} + c_2K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_2}{\partial x}\right] \\
&= c_1\frac{\partial}{\partial x}\left[K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_1}{\partial x}\right] + c_2\frac{\partial}{\partial x}\left[K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_2}{\partial x}\right] \\
&\neq c_1L(u_1) + c_2L(u_2)
\end{aligned}$$

Therefore, $L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is not a linear operator.