Exercise 2.2.4

In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$?

Solution

Part (a)

Here we have to show that

$$L(u_p + c_1u_1 + c_2u_2) = f.$$

Use the fact that L is a linear operator and simplify.

$$\begin{aligned} L(u_p + c_1u_1 + c_2u_2) &= L(u_p) + c_1L(u_1) + c_2L(u_2) \\ &= f + c_1(0) + c_2(0) \\ &= f \end{aligned}$$

Therefore, $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

Part (b)

u_{pi} is a particular solution corresponding to f_i, so we have the following equations to work with.

$$\begin{aligned} L(u_{p_1}) &= f_1 \\ L(u_{p_2}) &= f_2 \end{aligned}$$

Add these two equations to get

$$L(u_{p_1}) + L(u_{p_2}) = f_1 + f_2.$$

Use the fact that L is linear.

$$L(u_{p_1} + u_{p_2}) = f_1 + f_2$$

Therefore, $u = u_{p_1} + u_{p_2}$ is a particular solution for $f_1 + f_2$.