Exercise 2.2.5

If L is a linear operator, show that $L\left(\sum_{n=1}^{M} c_n u_n\right) = \sum_{n=1}^{M} c_n L(u_n)$. Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

Solution

Use mathematical induction to prove the result. Start by showing that the base case $M = 1$ results in a true statement.

$$L(c_1 u_1) = c_1 L(u_1)$$

Since L is a linear operator, this equation is true. Now assume that the inductive hypothesis,

$$L\left(\sum_{n=1}^{k} c_n u_n\right) = \sum_{n=1}^{k} c_n L(u_n),$$

is true and show that

$$L\left(\sum_{n=1}^{k+1} c_n u_n\right) = \sum_{n=1}^{k+1} c_n L(u_n).$$

We have

$$\begin{aligned} L\left(\sum_{n=1}^{k+1} c_n u_n\right) &= L\left(\sum_{n=1}^{k} c_n u_n + c_{k+1} u_{k+1}\right) \\ &= L\left(\sum_{n=1}^{k} c_n u_n\right) + L(c_{k+1} u_{k+1}). \\ &= \sum_{n=1}^{k} c_n L(u_n) + c_{k+1} L(u_{k+1}) \\ &= \sum_{n=1}^{k+1} c_n L(u_n) \end{aligned}$$

Therefore, by mathematical induction, $L\left(\sum_{n=1}^{M} c_n u_n\right) = \sum_{n=1}^{M} c_n L(u_n)$. Because u_1, u_2, $\ldots$, u_{M-1}, and u_M are solutions to a linear homogeneous equation, the following equations hold.

$$\begin{aligned} L(u_1) &= 0 \\ L(u_2) &= 0 \\ &\vdots \\ L(u_{M-1}) &= 0 \\ L(u_M) &= 0 \end{aligned}$$

Add all of them together to get

$$L(u_1) + L(u_2) + \cdots + L(u_{M-1}) + L(u_M) = 0.$$

Use the proven result to write the left side as

$$L(u_1 + u_2 + \cdots + u_{M-1} + u_M) = 0.$$

Therefore, $u = u_1 + u_2 + \cdots + u_{M-1} + u_M$ is a solution to the linear homogeneous equation as well, and the principle of superposition holds for any finite number of homogeneous solutions.