Exercise 2.3.10

For two- and three-dimensional vectors, the fundamental property of dot products, $\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta$, implies that

$$|\boldsymbol{A}\cdot\boldsymbol{B}| \le |\boldsymbol{A}||\boldsymbol{B}|. \tag{2.3.44}$$

In this exercise, we generalize this to n-dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\boldsymbol{A} - \gamma\boldsymbol{B}|^2 > 0$ implies (2.3.44), where $\gamma = \boldsymbol{A}\cdot\boldsymbol{B}/\boldsymbol{B}\cdot\boldsymbol{B}$.

(b) Express the inequality using both

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

(c) Generalize (2.3.44) to functions. [*Hint*: Let $\boldsymbol{A}\cdot\boldsymbol{B}$ mean the integral $\int_0^L A(x)B(x)\,dx$.]