Exercise 2.3.2

Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

(b) $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$ (If necessary, see Section 2.4.1.)

(d) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

(f) $\phi(a) = 0$ and $\phi(b) = 0$ (You may assume that $\lambda > 0$.)

(g) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + \phi(L) = 0$ (If necessary, see Section 5.8.)

Solution

Part (a)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0,\ \phi(\pi) = 0$$

Suppose first that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos\mu x + C_2 \sin\mu x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} \phi(0) &= C_1 = 0 \\ \phi(\pi) &= C_1 \cos\mu\pi + C_2 \sin\mu\pi = 0 \end{aligned}$$

The second equation reduces to $C_2 \sin\mu\pi = 0$. In order to avoid the trivial solution, we insist that $C_2 \neq 0$. Then

$$\begin{aligned} \sin\mu\pi &= 0 \\ \mu\pi &= n\pi, \quad n = 1, 2, \ldots \\ \mu_n &= n, \quad n = 1, 2, \ldots. \end{aligned}$$

Therefore, there are positive eigenvalues $\lambda_n = n^2$, and the eigenfunctions associated with them are

$$\begin{aligned}\phi(x) &= C_1 \cos \mu x + C_2 \sin \mu x \\ &= C_2 \sin \mu x \quad \rightarrow \quad \phi_n(x) = \sin nx.\end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{d\phi}{dx} = C_3$$

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\begin{aligned}\phi(0) &= C_4 = 0 \\ \phi(\pi) &= C_3\pi + C_4 = 0\end{aligned}$$

Since $C_4 = 0$, the second equation gives $C_3 = 0$. The trivial solution is obtained, which means that zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$.

$$\frac{d^2\phi}{dx^2} - \gamma^2\phi = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned}\phi(0) &= C_5 = 0 \\ \phi(\pi) &= C_5 \cosh \gamma\pi + C_6 \sinh \gamma\pi = 0\end{aligned}$$

The second equation reduces to $C_6 \sinh \gamma\pi = 0$. Because hyperbolic sine is not oscillatory, C_6 must be zero. This results in the trivial solution, which means that there are no negative eigenvalues.

Part (b)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0,\ \phi(1) = 0$$

Suppose first that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\phi(0) = C_1 = 0$$
$$\phi(1) = C_1 \cos \mu + C_2 \sin \mu = 0$$

The second equation reduces to $C_2 \sin \mu = 0$. In order to avoid the trivial solution, we insist that $C_2 \neq 0$. Then

$$\sin \mu = 0$$
$$\mu_n = n\pi, \quad n = 1, 2, \ldots.$$

Therefore, there are positive eigenvalues $\lambda_n = n^2\pi^2$, and the eigenfunctions associated with them are

$$\begin{aligned}\phi(x) &= C_1 \cos \mu x + C_2 \sin \mu x \\ &= C_2 \sin \mu x \quad \rightarrow \quad \phi_n(x) = \sin n\pi x.\end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{d\phi}{dx} = C_3$$

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\phi(0) = C_4 = 0$$
$$\phi(1) = C_3 + C_4 = 0$$

Since $C_4 = 0$, the second equation gives $C_3 = 0$. The trivial solution is obtained, which means that zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$.

$$\frac{d^2\phi}{dx^2} - \gamma^2\phi = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\phi(0) = C_5 = 0$$
$$\phi(1) = C_5 \cosh \gamma + C_6 \sinh \gamma = 0$$

The second equation reduces to $C_6 \sinh \gamma = 0$. Because hyperbolic sine is not oscillatory, C_6 must be zero. This results in the trivial solution, which means that there are no negative eigenvalues.

Part (c)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \frac{d\phi}{dx}(0) = 0, \; \frac{d\phi}{dx}(L) = 0$$

Suppose first that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos\mu x + C_2 \sin\mu x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \mu(-C_1 \sin\mu x + C_2 \cos\mu x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} \phi'(0) &= \mu(C_2) = 0 \quad \rightarrow \quad C_2 = 0 \\ \phi'(L) &= \mu(-C_1 \sin\mu L + C_2 \cos\mu L) = 0 \end{aligned}$$

The second equation reduces to $-C_1\mu \sin\mu L = 0$. In order to avoid the trivial solution, we insist that $C_1 \neq 0$. Then

$$\begin{aligned} -\mu \sin\mu L &= 0 \\ \sin\mu L &= 0 \\ \mu L &= n\pi, \quad n = 1, 2, \ldots \\ \mu_n &= \frac{n\pi}{L}, \quad n = 1, 2, \ldots. \end{aligned}$$

Therefore, there are positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} \phi(x) &= C_1 \cos\mu x + C_2 \sin\mu x \\ &= C_1 \cos\mu x \quad \rightarrow \quad \phi_n(x) = \cos\frac{n\pi x}{L}. \end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{d\phi}{dx} = C_3$$

C_3 is set to zero to satisfy the boundary conditions. Integrate once more.

$$\phi(x) = C_4$$

Zero is an eigenvalue because ϕ is nonzero; the eigenfunction associated with it is $\phi_0(x) = 1$. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$.

$$\frac{d^2\phi}{dx^2} - \gamma^2\phi = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \gamma(C_5 \sinh \gamma x + C_6 \cosh \gamma x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} \phi'(0) &= \gamma(C_6) = 0 \quad \rightarrow \quad C_6 = 0 \\ \phi'(L) &= \gamma(C_5 \sinh \gamma L + C_6 \cosh \gamma L) = 0 \end{aligned}$$

The second equation reduces to $C_5\gamma \sinh \gamma L = 0$. Because hyperbolic sine is not oscillatory, C_5 must be zero. This results in the trivial solution, which means that there are no negative eigenvalues.

Part (d)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0,\ \frac{d\phi}{dx}(L) = 0$$

Suppose first that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \mu(-C_1 \sin \mu x + C_2 \cos \mu x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} \phi(0) &= C_1 = 0 \\ \phi'(L) &= \mu(-C_1 \sin \mu L + C_2 \cos \mu L) = 0 \end{aligned}$$

The second equation reduces to $C_2\mu \cos \mu L = 0$. In order to avoid the trivial solution, we insist that $C_2 \neq 0$. Then

$$\begin{aligned} \mu \cos \mu L &= 0 \\ \cos \mu L &= 0 \\ \mu L &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots \\ \mu_n &= \frac{1}{2L}(2n-1)\pi, \quad n = 1, 2, \ldots. \end{aligned}$$

Therefore, there are positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/(4L^2)$, and the eigenfunctions associated with them are

$$\begin{aligned} \phi(x) &= C_1 \cos \mu x + C_2 \sin \mu x \\ &= C_2 \sin \mu x \quad \rightarrow \quad \phi_n(x) = \sin \frac{1}{2L}(2n-1)\pi x. \end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{d\phi}{dx} = C_3$$

C_3 is set to zero to satisfy $\phi'(L) = 0$. Integrate once more.

$$\phi(x) = C_4$$

C_4 is set to zero to satisfy $\phi(0) = 0$. This results in the trivial solution, which means that zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$.

$$\frac{d^2\phi}{dx^2} - \gamma^2\phi = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \gamma(C_5 \sinh \gamma x + C_6 \cosh \gamma x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} \phi(0) &= C_5 = 0 \\ \phi'(L) &= \gamma(C_5 \sinh \gamma L + C_6 \cosh \gamma L) = 0 \end{aligned}$$

The second equation reduces to $C_6\gamma \cosh \gamma L = 0$. Because hyperbolic cosine is not oscillatory, C_6 must be zero. This results in the trivial solution, which means that there are no negative eigenvalues.

Part (e)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \frac{d\phi}{dx}(0) = 0, \ \phi(L) = 0$$

Suppose first that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \mu(-C_1 \sin \mu x + C_2 \cos \mu x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned}\phi'(0) &= \mu(C_2) = 0 \quad \rightarrow \quad C_2 = 0 \\ \phi(L) &= C_1 \cos \mu L + C_2 \sin \mu L = 0\end{aligned}$$

The second equation reduces to $C_1 \cos \mu L = 0$. In order to avoid the trivial solution, we insist that $C_1 \neq 0$. Then

$$\begin{aligned}\cos \mu L &= 0 \\ \mu L &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots \\ \mu_n &= \frac{1}{2L}(2n-1)\pi, \quad n = 1, 2, \ldots.\end{aligned}$$

Therefore, there are positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/(4L^2)$, and the eigenfunctions associated with them are

$$\begin{aligned}\phi(x) &= C_1 \cos \mu x + C_2 \sin \mu x \\ &= C_1 \cos \mu x \quad \rightarrow \quad \phi_n(x) = \cos \frac{1}{2L}(2n-1)\pi x.\end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{d\phi}{dx} = C_3$$

C_3 is set to zero to satisfy $\phi'(0) = 0$. Integrate once more.

$$\phi(x) = C_4$$

C_4 is set to zero to satisfy $\phi(L) = 0$. This results in the trivial solution, which means that zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$.

$$\frac{d^2\phi}{dx^2} - \gamma^2\phi = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \gamma(C_5 \sinh \gamma x + C_6 \cosh \gamma x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned}\phi'(0) &= \gamma(C_6) = 0 \quad \rightarrow \quad C_6 = 0 \\ \phi(L) &= C_5 \cosh \gamma L + C_6 \sinh \gamma L = 0\end{aligned}$$

The second equation reduces to $C_5 \cosh \gamma L = 0$. Because hyperbolic cosine is not oscillatory, C_5 must be zero. This results in the trivial solution, which means that there are no negative eigenvalues.

Part (f)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(a) = 0, \ \phi(b) = 0$$

Suppose only that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\phi(a) = C_1 \cos \mu a + C_2 \sin \mu a = 0 \tag{1}$$

$$\phi(b) = C_1 \cos \mu b + C_2 \sin \mu b = 0 \tag{2}$$

Solve equation (1) for C_1.

$$C_1 \cos \mu a = -C_2 \sin \mu a \quad \rightarrow \quad C_1 = -C_2 \frac{\sin \mu a}{\cos \mu a}$$

Substitute this result for C_1 into equation (2).

$$\left(-C_2 \frac{\sin \mu a}{\cos \mu a}\right) \cos \mu b + C_2 \sin \mu b = 0$$

Assume that $C_2 \neq 0$ and divide both sides by $C_2 \cos \mu b$.

$$\left(-\frac{\sin \mu a}{\cos \mu a}\right) + \frac{\sin \mu b}{\cos \mu b} = 0$$

$$-\tan \mu a + \tan \mu b = 0$$

$$\tan \mu b = \tan \mu a$$

$$\mu b = \mu a + n\pi$$

$$\mu(b - a) = n\pi$$

$$\mu_n = \frac{n\pi}{b - a}, \quad n = 1, 2, \ldots$$

Note that n has the values it does because λ can't be zero, and negative values of n yield redundant values of λ. Therefore, there are positive eigenvalues $\lambda = n^2\pi^2/(b - a)^2$, and the eigenfunctions associated with them are

$$\begin{aligned}\phi(x) &= C_1 \cos \mu x + C_2 \sin \mu x \\ &= \left(-C_2 \frac{\sin \mu a}{\cos \mu a}\right) \cos \mu x + C_2 \sin \mu x \\ &= \frac{C_2}{\cos \mu a}(-\sin \mu a \cos \mu x + \sin \mu x \cos \mu a) \\ &= \frac{C_2}{\cos \mu a} \sin \mu(x-a) \\ &= C_3 \sin \mu(x-a) \quad \rightarrow \quad \phi_n(x) = \sin \frac{n\pi(x-a)}{b-a}.\end{aligned}$$

Part (g)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0, \ \frac{d\phi}{dx}(L) + \phi(L) = 0$$

Suppose first that λ is positive: $\lambda = \mu^2$.

$$\frac{d^2\phi}{dx^2} + \mu^2\phi = 0$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \mu(-C_1 \sin \mu x + C_2 \cos \mu x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned}\phi(0) &= C_1 = 0 \\ \phi'(L) + \phi(L) &= \mu(-C_1 \sin \mu L + C_2 \cos \mu L) + C_1 \cos \mu L + C_2 \sin \mu L = 0\end{aligned}$$

The second equation reduces to $C_2\mu \cos \mu L + C_2 \sin \mu L = 0$. In order to avoid the trivial solution, we insist that $C_2 \neq 0$. Then

$$\mu \cos \mu L + \sin \mu L = 0$$

$$\sin \mu L = -\mu \cos \mu L$$

$$\tan \mu_n L = -\mu_n, \quad n = 1, 2, \ldots.$$

Therefore, there are positive eigenvalues $\lambda_n = \mu_n^2$, and the eigenfunctions associated with them are

$$\begin{aligned}\phi(x) &= C_1 \cos \mu x + C_2 \sin \mu x \\ &= C_2 \sin \mu x \quad \rightarrow \quad \phi_n(x) = \sin \mu_n x.\end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{d\phi}{dx} = C_3$$

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\phi(0) = C_4 = 0$$
$$\phi'(L) + \phi(L) = C_3 + C_3 L + C_4 = 0$$

Since $C_4 = 0$, the second equation reduces to $C_3(1 + L) = 0$, so C_3 must be zero as well. This results in the trivial solution, which means that zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$.

$$\frac{d^2\phi}{dx^2} - \gamma^2 \phi = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \gamma(C_5 \sinh \gamma x + C_6 \cosh \gamma x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\phi(0) = C_5 = 0$$
$$\phi'(L) + \phi(L) = \gamma(C_5 \sinh \gamma L + C_6 \cosh \gamma L) + C_5 \cosh \gamma L + C_6 \sinh \gamma L = 0$$

The second equation reduces to $C_6 \gamma \cosh \gamma L + C_6 \sinh \gamma L = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\gamma \cosh \gamma L + \sinh \gamma L = 0$$

$$\sinh \gamma L = -\gamma \cosh \gamma L$$
$$\tanh \gamma L = -\gamma.$$

The graph of $\tanh \gamma L$ does not intersect $-\gamma$ at any nonzero value of γ, so there are no negative eigenvalues.