Exercise 2.3.6

Evaluate

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \qquad \text{for } n \geq 0,\ m \geq 0.$$

Use the trigonometric identity

$$\cos a\cos b = \frac{1}{2}[\cos(a+b)+\cos(a-b)].$$

(Be careful if $a-b=0$ or $a+b=0$.)

Solution

Assume that $n \neq m$. Use the product-to-sum formula for cosine-cosine.

$$\begin{aligned}
\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx &= \int_0^L \frac{1}{2}\left[\cos\left(\frac{n\pi x}{L}+\frac{m\pi x}{L}\right)+\cos\left(\frac{n\pi x}{L}-\frac{m\pi x}{L}\right)\right]dx \\
&= \frac{1}{2}\left[\int_0^L \cos\frac{(n+m)\pi x}{L}\,dx + \int_0^L \cos\frac{(n-m)\pi x}{L}\,dx\right] \\
&= \frac{1}{2}\left[\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\bigg|_{x=0}^{x=L} + \frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\bigg|_{x=0}^{x=L}\right] \\
&= \frac{1}{2}\left\{\frac{L}{(n+m)\pi}[\sin(n+m)\pi-\sin 0]+\frac{L}{(n-m)\pi}[\sin(n-m)\pi-\sin 0]\right\} \\
&= \frac{L}{2\pi}\left[\frac{\sin(n\pi+m\pi)}{n+m}+\frac{\sin(n\pi-m\pi)}{n-m}\right] \\
&= \frac{L}{2\pi}\left[\frac{(n-m)(\sin n\pi\cos m\pi+\sin m\pi\cos n\pi)+(n+m)(\sin n\pi\cos m\pi-\sin m\pi\cos n\pi)}{(n+m)(n-m)}\right] \\
&= \frac{L}{2\pi}\left(\frac{-2m\sin m\pi\cos n\pi+2n\sin n\pi\cos m\pi}{n^2-m^2}\right) \\
&= \frac{L}{\pi}\left(\frac{n\sin n\pi\cos m\pi-m\sin m\pi\cos n\pi}{n^2-m^2}\right)
\end{aligned}$$

If n and m are integers, then this integral is zero. Assume now that $n = m$.

$$\begin{aligned}
\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx &= \int_0^L \cos^2\frac{n\pi x}{L}\,dx \\
&= \int_0^L \frac{1}{2}\left(1+\cos\frac{2n\pi x}{L}\right)dx \\
&= \frac{1}{2}\left(\int_0^L dx + \int_0^L \cos\frac{2n\pi x}{L}\,dx\right) \\
&= \frac{1}{2}\left(L+\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\bigg|_{x=0}^{x=L}\right) \\
&= \frac{1}{2}\left[L+\frac{L}{2n\pi}(\sin 2n\pi-\sin 0)\right] \\
&= \frac{L}{2}\left(1+\frac{\sin 2n\pi}{2n\pi}\right)
\end{aligned}$$

If n is an integer, then this integral is $L/2$.