Exercise 2.3.7

Consider the following boundary value problem (if necessary, see Section 2.4.1):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0, \; \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint:* The answer is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \cos\frac{n\pi x}{L}.\Bigg]$$

What is λ_n?

(c) Show that the initial condition, $u(x,0) = f(x)$, is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

(d) Using Exercise 2.3.6, solve for A_0 and A_n $(n \geq 1)$.

(e) What happens to the temperature distribution as $t \to \infty$? Show that it approaches the steady-state temperature distribution (see Section 1.4).

Solution

Part (a)

The PDE is the governing equation for the temperature in a one-dimensional rod that is homogeneous and has constant cross-sectional area. The boundary conditions indicate that the rod is insulated at the $x = 0$ and $x = L$ ends. Initially the temperature distribution in the rod is $u(x,0) = f(x)$.

Part (b)

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \to \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,t) = 0 \quad &\to \quad X'(0)T(t) = 0 \quad \to \quad X'(0) = 0 \\
\frac{\partial u}{\partial x}(L,t) = 0 \quad &\to \quad X'(L)T(t) = 0 \quad \to \quad X'(L) = 0
\end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to x.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X'(L) &= \alpha(C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second equation reduces to $C_1\alpha \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, C_1 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{dX}{dx} = C_3$$

Apply the boundary conditions now.

$$X'(0) = C_3 = 0$$
$$X'(L) = C_3 = 0$$

Consequently,

$$\frac{dX}{dx} = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Zero is an eigenvalue because $X(x)$ is not zero. The eigenfunction associated with it is $X_0(x) = 1$. Solve the ODE for T now with $\lambda = 0$.

$$\frac{dT}{dt} = 0 \quad \rightarrow \quad T_0(t) = \text{constant}$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Take a derivative of it with respect to x.

$$X'(x) = \beta(-C_5 \sin \beta x + C_6 \cos \beta x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$X'(0) = \beta(C_6) = 0$$
$$X'(L) = \beta(-C_5 \sin \beta L + C_6 \cos \beta L) = 0$$

The first equation implies that $C_6 = 0$, so the second equation reduces to $-C_5 \beta \sin \beta L = 0$. To avoid the trivial solution, we insist that $C_5 \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta L &= 0 \\ \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \beta x + C_6 \sin \beta x \\ &= C_5 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{n^2\pi^2}{L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -\frac{kn^2\pi^2}{L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \quad \to \quad T_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\cos\frac{n\pi x}{L}$$

Part (c)

Apply the initial condition $u(x,0) = f(x)$ to determine A_0 and A_n.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x) \tag{1}$$

Part (d)

To find A_0, integrate both sides of equation (1) with respect to x from 0 to x.

$$\int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\right) dx = \int_0^L f(x)\,dx$$

Split up the integral on the left into two and bring the constants in front.

$$A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\frac{n\pi x}{L}\,dx}_{=\,0} = \int_0^L f(x)\,dx$$

Consequently,

$$A_0 L = \int_0^L f(x)\,dx$$

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx.}$$

To find A_n, multiply both sides of equation (1) by $\cos(m\pi x/L)$, where m is an integer,

$$A_0 \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = f(x)\cos\frac{m\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \left(A_0 \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right) dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Split up the integral on the left into two and bring the constants in front.

$$\underbrace{A_0 \int_0^L \cos\frac{m\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Because the cosine functions are orthogonal, the remaining integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$A_n \int_0^L \cos^2\frac{n\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Consequently,

$$A_n\left(\frac{L}{2}\right) = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.}$$

Part (e)

The temperature distribution approaches equilibrium as $t \to \infty$.

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left[A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\cos\frac{n\pi x}{L}\right] \\ &= A_0 \\ &= \frac{1}{L}\int_0^L f(x)\,dx\end{aligned}$$

In particular, the equilibrium temperature distribution is the average of the initial temperature distribution.