Exercise 2.3.10

For two- and three-dimensional vectors, the fundamental property of dot products, $\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta$, implies that

$$|\boldsymbol{A}\cdot\boldsymbol{B}| \le |\boldsymbol{A}||\boldsymbol{B}|. \tag{2.3.44}$$

In this exercise, we generalize this to n-dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\boldsymbol{A} - \gamma\boldsymbol{B}|^2 > 0$ implies (2.3.44), where $\gamma = \boldsymbol{A}\cdot\boldsymbol{B}/\boldsymbol{B}\cdot\boldsymbol{B}$.

(b) Express the inequality using both

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

(c) Generalize (2.3.44) to functions. [*Hint*: Let $\boldsymbol{A}\cdot\boldsymbol{B}$ mean the integral $\int_0^L A(x)B(x)\,dx$.]

Solution

Part (a)

Suppose that $|\boldsymbol{A} - \gamma\boldsymbol{B}|^2 > 0$.

$$\left|\boldsymbol{A} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right|^2 > 0$$

$$\left(\boldsymbol{A} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right)\cdot\left(\boldsymbol{A} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right) > 0$$

$$\boldsymbol{A}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left(-\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right) + \left(-\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right)\cdot\boldsymbol{A} + \left(-\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right)\cdot\left(-\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\right) > 0$$

$$\boldsymbol{A}\cdot\boldsymbol{A} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{A}\cdot\boldsymbol{B} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}\cdot\boldsymbol{A} + \frac{(\boldsymbol{A}\cdot\boldsymbol{B})^2}{(\boldsymbol{B}\cdot\boldsymbol{B})^2}\boldsymbol{B}\cdot\boldsymbol{B} > 0$$

$$\boldsymbol{A}\cdot\boldsymbol{A} - 2\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{A}\cdot\boldsymbol{B} + \frac{(\boldsymbol{A}\cdot\boldsymbol{B})^2}{(\boldsymbol{B}\cdot\boldsymbol{B})^2}\boldsymbol{B}\cdot\boldsymbol{B} > 0$$

$$\boldsymbol{A}\cdot\boldsymbol{A} - 2\frac{(\boldsymbol{A}\cdot\boldsymbol{B})^2}{\boldsymbol{B}\cdot\boldsymbol{B}} + \frac{(\boldsymbol{A}\cdot\boldsymbol{B})^2}{\boldsymbol{B}\cdot\boldsymbol{B}} > 0$$

$$\boldsymbol{A}\cdot\boldsymbol{A} - \frac{(\boldsymbol{A}\cdot\boldsymbol{B})^2}{\boldsymbol{B}\cdot\boldsymbol{B}} > 0$$

$$|\boldsymbol{A}|^2 - \frac{(\boldsymbol{A}\cdot\boldsymbol{B})^2}{|\boldsymbol{B}|^2} > 0$$

Multiply both sides by $|\boldsymbol{B}|^2$

$$|\boldsymbol{A}|^2|\boldsymbol{B}|^2 - (\boldsymbol{A}\cdot\boldsymbol{B})^2 > 0$$

$$(\boldsymbol{A}\cdot\boldsymbol{B})^2 < |\boldsymbol{A}|^2|\boldsymbol{B}|^2$$

Therefore, (noting that equality results if $\boldsymbol{B} = \boldsymbol{A}$)

$$|\boldsymbol{A}\cdot\boldsymbol{B}| \le |\boldsymbol{A}||\boldsymbol{B}|.$$

Part (b)

If $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n$, then

$$|\boldsymbol{A} \cdot \boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}| \quad \rightarrow \quad \left|\sum_{n=1}^{\infty} a_n b_n\right| \leq \left|\sqrt{\sum_{n=1}^{\infty} a_n^2}\right| \left|\sqrt{\sum_{n=1}^{\infty} b_n^2}\right|.$$

Since the square roots yield positive numbers, the absolute value signs can be removed.

$$\left|\sum_{n=1}^{\infty} a_n b_n\right| \leq \sqrt{\sum_{n=1}^{\infty} a_n^2} \sqrt{\sum_{n=1}^{\infty} b_n^2}$$

Part (c)

If $\boldsymbol{A} \cdot \boldsymbol{B} = \int_0^L A(x) B(x)\, dx$, then

$$|\boldsymbol{A} \cdot \boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}| \quad \rightarrow \quad \left|\int_0^L A(x) B(x)\, dx\right| \leq \left|\sqrt{\int_0^L [A(x)]^2\, dx}\right| \left|\sqrt{\int_0^L [B(x)]^2\, dx}\right|.$$

Since the square roots yield positive numbers, the absolute value signs can be removed.

$$\left|\int_0^L A(x) B(x)\, dx\right| \leq \sqrt{\int_0^L [A(x)]^2\, dx} \sqrt{\int_0^L [B(x)]^2\, dx}$$