Exercise 2.3.9

Redo Exercise 2.3.8 if $\alpha < 0$. [Be especially careful if $-\alpha/k = (n\pi/L)^2$.]

Solution

The initial boundary value problem considered in Exercise 2.3.8 was

$$\begin{aligned}&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u, \quad 0 < x < L,\ t > 0\\ &u(0,t) = 0\\ &u(L,t) = 0\\ &u(x,0) = f(x).\end{aligned}$$

Here it is assumed that $\alpha < 0$.

Part (a)

The equilibrium temperature distributions have no time dependence: $u_E = u_E(x)$. As a result, they satisfy

$$0 = k\frac{d^2 u_E}{dx^2} - \alpha u_E.$$

Divide both sides by k.

$$\frac{d^2 u_E}{dx^2} - \frac{\alpha}{k}u_E = 0$$

The general solution is written in terms of sine and cosine.

$$u_E(x) = C_1 \cos\sqrt{-\frac{\alpha}{k}}x + C_2 \sin\sqrt{-\frac{\alpha}{k}}x$$

Since the boundary conditions for u apply for all time, u_E satisfies the same conditions, $u_E(0) = 0$ and $u_E(L) = 0$. Apply them both to determine C_1 and C_2.

$$\begin{aligned}u_E(0) &= C_1 = 0\\ u_E(L) &= C_1 \cos\sqrt{-\frac{\alpha}{k}}L + C_2 \sin\sqrt{-\frac{\alpha}{k}}L = 0\end{aligned}$$

The second equation reduces to $C_2 \sin\sqrt{-\frac{\alpha}{k}}L = 0$. If it so happens that the argument of sine is a positive multiple of π,

$$\sqrt{-\frac{\alpha}{k}}L = n\pi, \quad n = 1, 2, \ldots,$$

then the equilibrium temperature distribution is

$$u_E(x) = C_2 \sin\frac{n\pi x}{L}.$$

Otherwise, the equilibrium temperature distribution is

$$u_E(x) = 0.$$

Part (b)

The PDE and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)] - \alpha[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned} u(0,t) &= 0 &\rightarrow&& X(0)T(t) &= 0 &\rightarrow&& X(0) &= 0 \\ u(L,t) &= 0 &\rightarrow&& X(L)T(t) &= 0 &\rightarrow&& X(L) &= 0 \end{aligned}$$

Separate variables in the PDE now.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2} - \alpha X(x)T(t)$$

Divide both sides by $kX(x)T(t)$.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k} = \lambda$$

As a result of using the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \mu^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \left(\frac{\alpha}{k} + \mu^2\right)X,$$

which only has a nontrivial solution if the quantity in parentheses is negative.

$$\frac{d^2X}{dx^2} = -\left(-\frac{\alpha}{k} - \mu^2\right)X$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_3\cos\sqrt{-\frac{\alpha}{k} - \mu^2}x + C_4\sin\sqrt{-\frac{\alpha}{k} - \mu^2}x$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_3 = 0 \\ X(L) &= C_3\cos\sqrt{-\frac{\alpha}{k} - \mu^2}L + C_4\sin\sqrt{-\frac{\alpha}{k} - \mu^2}L = 0 \end{aligned}$$

The second equation reduces to $C_4 \sin \sqrt{-\frac{\alpha}{k} - \mu^2} L = 0$. To avoid getting the trivial solution, we insist that $C_4 \neq 0$. Then

$$\begin{aligned}
\sin \sqrt{-\frac{\alpha}{k} - \mu^2} L &= 0 \\
\sqrt{-\frac{\alpha}{k} - \mu^2} L &= n\pi, \quad n = 1, 2, \ldots \\
\sqrt{-\frac{\alpha}{k} - \mu^2} &= \frac{n\pi}{L} \\
-\frac{\alpha}{k} - \mu^2 &= \frac{n^2\pi^2}{L^2} \\
\mu^2 &= -\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}.
\end{aligned}$$

The number of positive eigenvalues is constrained by the fact that $\mu^2 > 0$.

$$\begin{gathered}
-\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2} > 0 \\
\frac{n^2\pi^2}{L^2} < -\frac{\alpha}{k} \\
n^2 < -\frac{\alpha}{k}\frac{L^2}{\pi^2} \\
0 < n < \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}
\end{gathered}$$

Consequently, the positive eigenvalues are $\lambda = -\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}$ for $0 < n < \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}$, and the eigenfunctions associated with them are

$$\begin{aligned}
X(x) &= C_3 \cos \sqrt{-\frac{\alpha}{k} - \mu^2} x + C_4 \sin \sqrt{-\frac{\alpha}{k} - \mu^2} x \\
&= C_4 \sin \sqrt{-\frac{\alpha}{k} - \mu^2} x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{L}.
\end{aligned}$$

Now solve the ODE for T with this formula for λ.

$$\frac{dT}{dt} = k\left(-\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}\right)T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_5 \exp\left[k\left(-\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}\right)t\right] \quad \rightarrow \quad T_n(t) = \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right]$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \frac{\alpha}{k}X,$$

which is the same as the one for $u_E(x)$. Zero is an eigenvalue if it so happens that

$$\sqrt{-\frac{\alpha}{k}}L = n\pi \quad \text{or} \quad n = \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}, \quad n = 1, 2, \ldots.$$

The eigenfunction associated with it is $X_0(x) = \sin \frac{n\pi x}{L}$. Now solve the ODE for T with $\lambda = 0$.

$$\frac{dT}{dt} = 0 \quad \rightarrow \quad T = \text{constant}$$

Suppose thirdly that λ is negative: $\lambda = -\gamma^2$. The ODE for X becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k} = -\gamma^2 \quad \rightarrow \quad \frac{d^2X}{dx^2} = -\left(-\frac{\alpha}{k} + \gamma^2\right)X$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_6 \cos\sqrt{-\frac{\alpha}{k} + \gamma^2}x + C_7 \sin\sqrt{-\frac{\alpha}{k} + \gamma^2}x$$

Apply the boundary conditions to determine C_6 and C_7.

$$X(0) = C_6 = 0$$
$$X(L) = C_6 \cos\sqrt{-\frac{\alpha}{k} + \gamma^2}L + C_7 \sin\sqrt{-\frac{\alpha}{k} + \gamma^2}L = 0$$

The second equation reduces to $C_7 \sin\sqrt{-\frac{\alpha}{k} + \gamma^2}L = 0$. To avoid getting the trivial solution, we insist that $C_7 \neq 0$. Then

$$\sin\sqrt{-\frac{\alpha}{k} + \gamma^2}L = 0$$
$$\sqrt{-\frac{\alpha}{k} + \gamma^2}L = n\pi, \quad n = 1, 2, \ldots$$
$$\sqrt{-\frac{\alpha}{k} + \gamma^2} = \frac{n\pi}{L}$$
$$-\frac{\alpha}{k} + \gamma^2 = \frac{n^2\pi^2}{L^2}$$
$$\gamma^2 = \frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}$$

The number of negative eigenvalues is constrained by the fact that $\gamma^2 > 0$.

$$\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2} > 0$$
$$\frac{n^2\pi^2}{L^2} > -\frac{\alpha}{k}$$
$$n^2 > -\frac{\alpha}{k}\frac{L^2}{\pi^2}$$
$$\sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} < n < \infty$$

Consequently, the negative eigenvalues are $\lambda = -\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}$ for $\sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} < n < \infty$. The eigenfunctions associated with them are

$$X(x) = C_6 \cos\sqrt{-\frac{\alpha}{k} + \gamma^2}x + C_7 \sin\sqrt{-\frac{\alpha}{k} + \gamma^2}x$$
$$= C_7 \sin\sqrt{-\frac{\alpha}{k} + \gamma^2}x \quad \rightarrow \quad X_n(x) = \sin\frac{n\pi x}{L}.$$

Now solve the ODE for T with this formula for λ.

$$\frac{dT}{dt} = k\left(-\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}\right)T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_8 \exp\left[k\left(-\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}\right)t\right] \quad \rightarrow \quad T_n(t) = \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right]$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over the eigenvalues. Depending what $\sqrt{-\alpha/k}(L/\pi)$ is, the eigenvalues (and hence the solution) will be different.

Therefore,

$$u(x,t)=\begin{cases}\displaystyle\sum_{n=1}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}<1\\[2ex] \displaystyle B_0\sin\frac{\pi x}{L}+\sum_{n=2}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}=1\\[2ex] \displaystyle\sum_{0<n<\sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L}+B_0\sin\sqrt{-\frac{\alpha}{k}}x & \\ \displaystyle\qquad+\sum_{\sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}<n<\infty}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}>1 \text{ and } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}\in\mathbb{Z}^+\\[2ex] \displaystyle\sum_{0<n<\sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L} & \\ \displaystyle\qquad+\sum_{\sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}<n<\infty}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}>1 \text{ and } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}\notin\mathbb{Z}^+\end{cases}.$$

The sums in blue are linear combinations over the negative eigenvalues. The exponential functions in them tend to zero as $t\to\infty$. On the other hand, the sums in red are linear combinations over the positive eigenvalues. The exponential functions in them tend to ∞ as $t\to\infty$. As a result,

$$\lim_{t\to\infty}u(x,t)=\begin{cases}0 & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}<1\\ B_0\sin\frac{\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}=1\\ \infty & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi}>1\end{cases}.$$

The solution can be written compactly as

$$u(x,t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right] \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} < 1 \\ \\ \displaystyle B_0 \sin\frac{\pi x}{L} + \sum_{n=2}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right] \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} = 1 \\ \\ \displaystyle B_0 \sin\frac{p\pi x}{L} + \sum_{\substack{n=1 \\ n\neq p}}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right] \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} > 1 \text{ and } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} = p \in \mathbb{Z}^+ \\ \\ \displaystyle\sum_{n=1}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right] \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} > 1 \text{ and } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} \notin \mathbb{Z}^+ \end{cases}.$$

The final task is to use the initial condition $u(x,0) = f(x)$ to determine the coefficients.

$$u(x,0) = f(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} < 1 \\ \\ \displaystyle B_0 \sin\frac{\pi x}{L} + \sum_{n=2}^{\infty} B_n \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} = 1 \\ \\ \displaystyle B_0 \sin\frac{p\pi x}{L} + \sum_{\substack{n=1 \\ n\neq p}}^{\infty} B_n \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} > 1 \text{ and } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} = p \in \mathbb{Z}^+ \\ \\ \displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} & \text{if } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} > 1 \text{ and } \sqrt{-\frac{\alpha}{k}}\frac{L}{\pi} \notin \mathbb{Z}^+ \end{cases}$$

Each of these cases is a Fourier sine series expansion of $f(x)$. The coefficients are therefore

$$\begin{cases} B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots & \text{if } \sqrt{-\dfrac{\alpha}{k}}\dfrac{L}{\pi} < 1 \\[2em] B_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{\pi x}{L}\,dx,\ B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 2, 3, \ldots & \text{if } \sqrt{-\dfrac{\alpha}{k}}\dfrac{L}{\pi} = 1 \\[2em] B_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{p\pi x}{L}\,dx,\ B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad \begin{matrix} n = 1, 2, \ldots \\ n \neq p \end{matrix} & \text{if } \sqrt{-\dfrac{\alpha}{k}}\dfrac{L}{\pi} > 1 \text{ and } \sqrt{-\dfrac{\alpha}{k}}\dfrac{L}{\pi} = p \in \mathbb{Z}^+ \\[2em] B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots & \text{if } \sqrt{-\dfrac{\alpha}{k}}\dfrac{L}{\pi} > 1 \text{ and } \sqrt{-\dfrac{\alpha}{k}}\dfrac{L}{\pi} \notin \mathbb{Z}^+ \end{cases}$$