Exercise 2.4.3

Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

subject to

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

Solution

Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE becomes

$$\frac{d^2\phi}{dx^2} = -\alpha^2\phi.$$

The general solution is written in terms of sine and cosine.

$$\phi(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

Take a derivative of it with respect to x.

$$\phi'(x) = \alpha(-C_1 \sin \alpha x + C_2 \cos \alpha x)$$

Apply the boundary conditions to obtain a system of equations involving C_1 and C_2.

$$\begin{aligned} \phi(0) = C_1 &= C_1 \cos 2\pi\alpha + C_2 \sin 2\pi\alpha = \phi(2\pi) \\ \phi'(0) = \alpha(C_2) &= \alpha(-C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha) = \phi'(2\pi) \end{aligned}$$

$$\begin{cases} C_1 = C_1 \cos 2\pi\alpha + C_2 \sin 2\pi\alpha \\ C_2 = -C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha \end{cases}$$

$$\begin{cases} C_1(1 - \cos 2\pi\alpha) = C_2 \sin 2\pi\alpha \\ C_2(1 - \cos 2\pi\alpha) = -C_1 \sin 2\pi\alpha \end{cases}$$

Both equations are satisfied if $\alpha = n$, where $n = 1, 2, \ldots$. The positive eigenvalues are $\lambda = n^2$, and the eigenfunctions associated with them are

$$\phi(x) = C_1 \cos \alpha x + C_2 \sin \alpha x \quad \rightarrow \quad \boxed{\phi_n(x) = C_1 \cos nx + C_2 \sin nx.}$$

n only takes on the values it does because negative integers result in redundant values for λ.
Suppose secondly that λ is zero: $\lambda = 0$. The ODE for ϕ becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

Integrate both sides with respect to x.

$$\frac{d\phi}{dx} = C_3$$

Apply the second boundary condition to determine C_3.

$$\phi'(0) = C_3 = C_3 = \phi'(2\pi)$$

C_3 remains arbitrary. Integrate both sides of the previous equation with respect to x once more.

$$\phi(x) = C_3x + C_4$$

Apply the first boundary condition.

$$\phi(0) = C_4 = 2\pi C_3 + C_4 = \phi(2\pi)$$

$C_4 = 2\pi C_3 + C_4$ leads to $C_3 = 0$.

$$\phi(x) = C_4 \quad \rightarrow \quad \boxed{\phi_0(x) = 1}$$

Because $\phi(x)$ is a nontrivial function, zero is an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for ϕ becomes

$$\frac{d^2\phi}{dx^2} = \beta^2\phi.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_5 \cosh \beta x + C_6 \sinh \beta x$$

Take a derivative of it.

$$\phi'(x) = \beta(C_5 \sinh \beta x + C_6 \cosh \beta x)$$

Apply the boundary conditions to determine C_5 and C_6.

$$\phi(0) = C_5 = C_5 \cosh 2\pi\beta + C_6 \sinh 2\pi\beta = \phi(2\pi)$$
$$\phi'(0) = \beta(C_6) = \beta(C_5 \sinh 2\pi\beta + C_6 \cosh 2\pi\beta) = \phi'(2\pi)$$

$$\begin{cases} C_5 = C_5 \cosh 2\pi\beta + C_6 \sinh 2\pi\beta \\ C_6 = C_5 \sinh 2\pi\beta + C_6 \cosh 2\pi\beta \end{cases}$$

$$\begin{cases} C_5(1 - \cosh 2\pi\beta) = C_6 \sinh 2\pi\beta \\ C_6(1 - \cosh 2\pi\beta) = C_5 \sinh 2\pi\beta \end{cases}$$

No nonzero value of β satisfies these equations, so $C_5 = 0$ and $C_6 = 0$. The trivial solution $\phi(x) = 0$ is obtained, which means there are no negative eigenvalues.