Exercise 2.4.4

Explicitly show that there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{subject to} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0.$$

Solution

Suppose that λ is negative: $\lambda = -\beta^2$. Then the ODE for ϕ becomes

$$\frac{d^2\phi}{dx^2} = \beta^2\phi.$$

The general solution is written in terms of hyperbolic cosine and hyperbolic sine.

$$\phi(x) = C_1 \cosh \beta x + C_2 \sinh \beta x$$

Take a derivative of it.

$$\phi'(x) = \beta(C_1 \sinh \beta x + C_2 \cosh \beta x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} \phi'(0) &= \beta(C_2) = 0 \\ \phi'(L) &= \beta(C_1 \sinh \beta L + C_2 \cosh \beta L) = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, which means the second equation reduces to $\beta(C_1 \sinh \beta L) = 0$. Because hyperbolic sine is not oscillatory, the only way $\beta(C_1 \sinh \beta L) = 0$ is satisfied is if $C_1 = 0$. The trivial solution $\phi(x) = 0$ is obtained, so there are no negative eigenvalues.