Exercise 2.5.1

Solve Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in y, let $u(x,y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in x, let $u(x,y) = \phi(x)h(y)$.]:

(a) $\frac{\partial u}{\partial x}(0,y) = 0,\quad \frac{\partial u}{\partial x}(L,y) = 0,\quad u(x,0) = 0,\quad u(x,H) = f(x)$

(b) $\frac{\partial u}{\partial x}(0,y) = g(y),\quad \frac{\partial u}{\partial x}(L,y) = 0,\quad u(x,0) = 0,\quad u(x,H) = 0$

(c) $\frac{\partial u}{\partial x}(0,y) = 0,\quad u(L,y) = g(y),\quad u(x,0) = 0,\quad u(x,H) = 0$

(d) $u(0,y) = g(y),\quad u(L,y) = 0,\quad \frac{\partial u}{\partial y}(x,0) = 0,\quad u(x,H) = 0$

(e) $u(0,y) = 0,\quad u(L,y) = 0,\quad u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0,\quad u(x,H) = f(x)$

(f) $u(0,y) = f(y),\quad u(L,y) = 0,\quad \frac{\partial u}{\partial y}(x,0) = 0,\quad \frac{\partial u}{\partial y}(x,H) = 0$

(g) $\frac{\partial u}{\partial x}(0,y) = 0,\quad \frac{\partial u}{\partial x}(L,y) = 0,\quad u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases},\quad \frac{\partial u}{\partial y}(x,H) = 0$

(h) $u(0,y) = 0,\quad u(L,y) = g(y),\quad u(x,0) = 0,\quad u(x,H) = 0$

Solution

Part (a)

$$\begin{aligned}
&\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leq x \leq L,\ 0 \leq y \leq H \\
&\frac{\partial u}{\partial x}(0,y) = 0 \\
&\frac{\partial u}{\partial x}(L,y) = 0 \\
&u(x,0) = 0 \\
&u(x,H) = f(x)
\end{aligned}$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x,y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \to \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{array}{lllll} \frac{\partial u}{\partial x}(0,y) = 0 & \rightarrow & X'(0)Y(y) = 0 & \rightarrow & X'(0) = 0 \\ \frac{\partial u}{\partial x}(L,y) = 0 & \rightarrow & X'(L)Y(y) = 0 & \rightarrow & X'(L) = 0 \\ u(x,0) = 0 & \rightarrow & X(x)Y(0) = 0 & \rightarrow & Y(0) = 0 \end{array}$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for X first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$X'' = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative of it.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X'(L) &= \alpha(C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0 \end{aligned}$$

The first equation implies that C_2, so the second one reduces to $C_1\alpha \sinh \alpha L = 0$. No nonzero value of α satisfies this equation, so C_1 must be zero. The trivial solution is obtained, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_3$$

Apply the boundary conditions to determine C_3.

$$\begin{aligned} X'(0) &= C_3 = 0 \\ X'(L) &= C_3 = 0 \end{aligned}$$

Consequently,

$$X' = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Because $X(x)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $X_0(x) = 1$. With this value for λ, solve the ODE for Y.

$$Y'' = 0$$

Integrate both sides with respect to y twice.

$$Y(y) = C_5 y + C_6$$

Apply the boundary condition to determine one of the constants.

$$Y(0) = C_6 = 0$$

So then

$$Y(y) = C_5 y.$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Take a derivative of it.

$$X'(x) = \beta(-C_7 \sin \beta x + C_8 \cos \beta x)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X'(0) &= \beta(C_8) = 0 \\ X'(L) &= \beta(-C_7 \sin \beta L + C_8 \cos \beta L) = 0 \end{aligned}$$

The first equation implies that $C_8 = 0$, so the second one reduces to $-C_7\beta \sin \beta L = 0$. To avoid getting the trivial solution, we insist that $C_7 \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta L &= 0 \\ \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \beta x + C_8 \sin \beta x \\ &= C_7 \cos \beta x \quad \to \quad X_n(x) = \cos \frac{n\pi x}{L}. \end{aligned}$$

With this formula for λ, solve the ODE for Y now.

$$\frac{d^2Y}{dy^2} = \frac{n^2\pi^2}{L^2} Y$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_9 \cosh \frac{n\pi y}{L} + C_{10} \sinh \frac{n\pi y}{L}$$

Use the boundary condition to determine one of the constants.

$$Y(0) = C_9 = 0$$

So then

$$Y(y) = C_{10} \sinh \frac{n\pi y}{L}.$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}}$$

Use the final inhomogeneous boundary condition $u(x, H) = f(x)$ to determine A_0 and A_n.

$$u(x, H) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi H}{L} \cos \frac{n\pi x}{L} = f(x) \tag{1}$$

To find A_0, integrate both sides of equation (1) with respect to x from 0 to L.

$$\int_0^L \left(A_0 H + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi H}{L} \cos \frac{n\pi x}{L} \right) dx = \int_0^L f(x)\, dx$$

Split up the integral on the left and bring the constants in front.

$$A_0 H \int_0^L dx + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi H}{L} \underbrace{\int_0^L \cos \frac{n\pi x}{L}\, dx}_{=\,0} = \int_0^L f(x)\, dx$$

$$A_0 HL = \int_0^L f(x)\,dx$$

So then

$$\boxed{A_0 = \frac{1}{HL}\int_0^L f(x)\,dx.}$$

To find A_n, multiply both sides of equation (1) by $\cos(m\pi x/L)$, where m is an integer,

$$A_0 H \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = f(x)\cos\frac{m\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \left(A_0 H \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right)dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Split up the integral on the left and bring the constants in front.

$$A_0 H \underbrace{\int_0^L \cos\frac{m\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi H}{L}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Because the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$A_n \sinh\frac{n\pi H}{L}\int_0^L \cos^2\frac{n\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$A_n \sinh\frac{n\pi H}{L}\left(\frac{L}{2}\right) = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

So then

$$\boxed{A_n = \frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.}$$

Part (b)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$

$$\frac{\partial u}{\partial x}(0, y) = g(y)$$

$$\frac{\partial u}{\partial x}(L, y) = 0$$

$$u(x, 0) = 0$$

$$u(x, H) = 0$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \to \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial x}(L, y) = 0 \quad &\to \quad X'(L)Y(y) = 0 \quad &\to \quad X'(L) = 0 \\ u(x, 0) = 0 \quad &\to \quad X(x)Y(0) = 0 \quad &\to \quad Y(0) = 0 \\ u(x, H) = 0 \quad &\to \quad X(x)Y(H) = 0 \quad &\to \quad Y(H) = 0 \end{aligned}$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for Y first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Y becomes

$$Y'' = -\alpha^2 Y.$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_1 \cos \alpha y + C_2 \sin \alpha y$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} Y(0) &= C_1 = 0 \\ Y(H) &= C_1 \cos \alpha H + C_2 \sin \alpha H = 0 \end{aligned}$$

The second equation reduces to $C_2 \sin \alpha H = 0$. To avoid getting the trivial solution, we insist that $C_2 \neq 0$. Then

$$\begin{aligned} \sin \alpha H &= 0 \\ \alpha H &= n\pi, \quad n = 1, 2, \ldots \\ \alpha_n &= \frac{n\pi}{H}. \end{aligned}$$

There are positive eigenvalues $\lambda = n^2\pi^2/H^2$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_1 \cos \alpha y + C_2 \sin \alpha y \\ &= C_2 \sin \alpha y \quad \rightarrow \quad Y_n(y) = \sin \frac{n\pi y}{H}. \end{aligned}$$

With this formula for λ, the ODE for X becomes

$$\frac{d^2 X}{dx^2} = \frac{n^2\pi^2}{H^2} X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \frac{n\pi x}{H} + C_4 \sinh \frac{n\pi x}{H}$$

Take a derivative of it.

$$X'(x) = \frac{n\pi}{H} \left(C_3 \sinh \frac{n\pi x}{H} + C_4 \cosh \frac{n\pi x}{H} \right)$$

Apply the boundary condition to determine one of the constants.

$$X'(L) = \frac{n\pi}{H} \left(C_3 \sinh \frac{n\pi L}{H} + C_4 \cosh \frac{n\pi L}{H} \right) = 0 \quad \rightarrow \quad C_4 = -C_3 \frac{\sinh \frac{n\pi L}{H}}{\cosh \frac{n\pi L}{H}}$$

So then

$$\begin{aligned} X(x) &= C_3 \cosh \frac{n\pi x}{H} - C_3 \frac{\sinh \frac{n\pi L}{H}}{\cosh \frac{n\pi L}{H}} \sinh \frac{n\pi x}{H} \\ &= \frac{C_3}{\cosh \frac{n\pi L}{H}} \left(\cosh \frac{n\pi x}{H} \cosh \frac{n\pi L}{H} - \sinh \frac{n\pi L}{H} \sinh \frac{n\pi x}{H} \right) \\ &= \frac{C_3}{\cosh \frac{n\pi L}{H}} \cosh \left[\frac{n\pi}{H}(x - L) \right] \quad \rightarrow \quad X_n(x) = \cosh \left[\frac{n\pi}{H}(x - L) \right]. \end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Y becomes

$$Y'' = 0.$$

Integrate both sides with respect to y twice.

$$Y(y) = C_5 y + C_6$$

Apply the boundary conditions to determine C_5 and C_6.

$$Y(0) = C_6 = 0$$
$$Y(H) = C_5 H + C_6 = 0$$

The second equation reduces to $C_5 H = 0$, which means $C_5 = 0$. The trivial solution $Y(y) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Y becomes

$$Y'' = \beta^2 Y.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_7 \cosh \beta y + C_8 \sinh \beta y$$

Apply the boundary conditions to determine C_7 and C_8.

$$Y(0) = C_7 = 0$$
$$Y(H) = C_7 \cosh \beta H + C_8 \sinh \beta H = 0$$

The second equation reduces to $C_8 \sinh \beta H = 0$. No nonzero value of β can satisfy this equation, so C_8 must be zero. The trivial solution $Y(y) = 0$ is obtained, which means there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \cosh\left[\frac{n\pi}{H}(x - L)\right] \sin \frac{n\pi y}{H}}$$

Use the remaining inhomogeneous boundary condition $\frac{\partial u}{\partial x}(0, y) = g(y)$ to determine B_n. Take a derivative of the general solution with respect to x.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh\left[\frac{n\pi}{H}(x - L)\right] \sin \frac{n\pi y}{H}$$

Apply the boundary condition.

$$\frac{\partial u}{\partial x}(0, y) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh\left[\frac{n\pi}{H}(-L)\right] \sin \frac{n\pi y}{H} = g(y)$$

$$\sum_{n=1}^{\infty} \left(-B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H}\right) \sin \frac{n\pi y}{H} = g(y)$$

To find B_n, multiply both sides by $\sin(m\pi y/H)$, where m is an integer,

$$\sum_{n=1}^{\infty}\left(-B_n\frac{n\pi}{H}\sinh\frac{n\pi L}{H}\right)\sin\frac{n\pi y}{H}\sin\frac{m\pi y}{H} = g(y)\sin\frac{m\pi y}{H}$$

and then integrate both sides with respect to y from 0 to H.

$$\int_0^H \sum_{n=1}^{\infty}\left(-B_n\frac{n\pi}{H}\sinh\frac{n\pi L}{H}\right)\sin\frac{n\pi y}{H}\sin\frac{m\pi y}{H}\,dy = \int_0^H g(y)\sin\frac{m\pi y}{H}\,dy$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty}\left(-B_n\frac{n\pi}{H}\sinh\frac{n\pi L}{H}\right)\int_0^H \sin\frac{n\pi y}{H}\sin\frac{m\pi y}{H}\,dy = \int_0^H g(y)\sin\frac{m\pi y}{H}\,dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$\left(-B_n\frac{n\pi}{H}\sinh\frac{n\pi L}{H}\right)\int_0^H \sin^2\frac{n\pi y}{H}\,dy = \int_0^H g(y)\sin\frac{n\pi y}{H}\,dy$$

$$\left(-B_n\frac{n\pi}{H}\sinh\frac{n\pi L}{H}\right)\left(\frac{H}{2}\right) = \int_0^H g(y)\sin\frac{n\pi y}{H}\,dy$$

So then

$$\boxed{B_n = -\frac{2}{n\pi\sinh\frac{n\pi L}{H}}\int_0^H g(y)\sin\frac{n\pi y}{H}\,dy.}$$

Part (c)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$
$$\frac{\partial u}{\partial x}(0, y) = 0$$
$$u(L, y) = g(y)$$
$$u(x, 0) = 0$$
$$u(x, H) = 0$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \to \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial x}(0, y) = 0 \quad &\to \quad X'(0)Y(y) = 0 \quad \to \quad X'(0) = 0 \\ u(x, 0) = 0 \quad &\to \quad X(x)Y(0) = 0 \quad \to \quad Y(0) = 0 \\ u(x, H) = 0 \quad &\to \quad X(x)Y(H) = 0 \quad \to \quad Y(H) = 0 \end{aligned}$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for Y first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Y becomes

$$Y'' = -\alpha^2 Y.$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_1 \cos \alpha y + C_2 \sin \alpha y$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} Y(0) &= C_1 = 0 \\ Y(H) &= C_1 \cos \alpha H + C_2 \sin \alpha H = 0 \end{aligned}$$

The second equation reduces to $C_2 \sin \alpha H = 0$. To avoid getting the trivial solution, we insist that $C_2 \neq 0$. Then

$$\begin{aligned} \sin \alpha H &= 0 \\ \alpha H &= n\pi, \quad n = 1, 2, \ldots \\ \alpha_n &= \frac{n\pi}{H}. \end{aligned}$$

There are positive eigenvalues $\lambda = n^2\pi^2/H^2$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_1 \cos \alpha y + C_2 \sin \alpha y \\ &= C_2 \sin \alpha y \quad \rightarrow \quad Y_n(y) = \sin \frac{n\pi y}{H}. \end{aligned}$$

With this formula for λ, the ODE for X becomes

$$\frac{d^2 X}{dx^2} = \frac{n^2\pi^2}{H^2} X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \frac{n\pi x}{H} + C_4 \sinh \frac{n\pi x}{H}$$

Take a derivative of it.

$$X'(x) = \frac{n\pi}{H}\left(C_3 \sinh \frac{n\pi x}{H} + C_4 \cosh \frac{n\pi x}{H}\right)$$

Apply the boundary condition to determine one of the constants.

$$X'(0) = \frac{n\pi}{H}(C_4) = 0 \quad \rightarrow \quad C_4 = 0$$

So then

$$X(x) = C_3 \cosh \frac{n\pi x}{H} \quad \rightarrow \quad X_n(x) = \cosh \frac{n\pi x}{H}.$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Y becomes

$$Y'' = 0.$$

Integrate both sides with respect to y twice.

$$Y(y) = C_5 y + C_6$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} Y(0) &= C_6 = 0 \\ Y(H) &= C_5 H + C_6 = 0 \end{aligned}$$

The second equation reduces to $C_5 H = 0$, which means $C_5 = 0$. The trivial solution $Y(y) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Y becomes

$$Y'' = \beta^2 Y.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_7 \cosh \beta y + C_8 \sinh \beta y$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} Y(0) &= C_7 = 0 \\ Y(H) &= C_7 \cosh \beta H + C_8 \sinh \beta H = 0 \end{aligned}$$

The second equation reduces to $C_8 \sinh \beta H = 0$. No nonzero value of β can satisfy this equation, so C_8 must be zero. The trivial solution $Y(y) = 0$ is obtained, which means there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}}$$

Use the remaining inhomogeneous boundary condition $u(L, y) = g(y)$ to determine B_n.

$$u(L, y) = \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} = g(y)$$

Multiply both sides by $\sin(m\pi y/H)$, where m is an integer,

$$\sum_{n=1}^{\infty} B_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} \sin \frac{m\pi y}{H} = g(y) \sin \frac{m\pi y}{H}$$

and then integrate both sides with respect to y from 0 to H.

$$\int_0^H \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} \sin \frac{m\pi y}{H} \, dy = \int_0^H g(y) \sin \frac{m\pi y}{H} \, dy$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} B_n \cosh \frac{n\pi L}{H} \int_0^H \sin \frac{n\pi y}{H} \sin \frac{m\pi y}{H} \, dy = \int_0^H g(y) \sin \frac{m\pi y}{H} \, dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$B_n \cosh \frac{n\pi L}{H} \int_0^H \sin^2 \frac{n\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$$

$$B_n \cosh \frac{n\pi L}{H} \left(\frac{H}{2}\right) = \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$$

So then

$$\boxed{B_n = \frac{2}{H \cosh \frac{n\pi L}{H}} \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy.}$$

Part (d)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$
$$u(0, y) = g(y)$$
$$u(L, y) = 0$$
$$\frac{\partial u}{\partial y}(x, 0) = 0$$
$$u(x, H) = 0$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u(L, y) = 0 \quad &\rightarrow \quad X(L)Y(y) = 0 \quad &\rightarrow \quad X(L) = 0 \\ \frac{\partial u}{\partial y}(x, 0) = 0 \quad &\rightarrow \quad X(x)Y'(0) = 0 \quad &\rightarrow \quad Y'(0) = 0 \\ u(x, H) = 0 \quad &\rightarrow \quad X(x)Y(H) = 0 \quad &\rightarrow \quad Y(H) = 0 \end{aligned}$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called eigenvalues, and the solutions themselves are known as eigenfunctions. We will solve the ODE for Y first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Y becomes

$$Y'' = -\alpha^2 Y.$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_1 \cos \alpha y + C_2 \sin \alpha y$$

Take a derivative of it.

$$Y'(y) = \alpha(-C_1 \sin \alpha y + C_2 \cos \alpha y)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} Y'(0) &= \alpha(C_2) = 0 \\ Y(H) &= C_1 \cos \alpha H + C_2 \sin \alpha H = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second one reduces to $C_1 \cos \alpha H = 0$. To avoid getting the trivial solution, we insist that $C_1 \neq 0$. Then

$$\begin{aligned} \cos \alpha H &= 0 \\ \alpha H &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots \\ \alpha_n &= \frac{1}{2H}(2n-1)\pi. \end{aligned}$$

There are positive eigenvalues $\lambda = (2n-1)^2\pi^2/(4H^2)$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_1 \cos \alpha y + C_2 \sin \alpha y \\ &= C_1 \cos \alpha y \quad \rightarrow \quad Y_n(y) = \cos \frac{(2n-1)\pi y}{2H}. \end{aligned}$$

With this formula for λ, the ODE for X becomes

$$\frac{d^2X}{dx^2} = \frac{(2n-1)^2\pi^2}{4H^2} X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \frac{(2n-1)\pi x}{2H} + C_4 \sinh \frac{(2n-1)\pi x}{2H}$$

Apply the boundary condition to determine one of the constants.

$$X(L) = C_3 \cosh \frac{(2n-1)\pi L}{2H} + C_4 \sinh \frac{(2n-1)\pi L}{2H} = 0 \quad \rightarrow \quad C_3 = -C_4 \frac{\sinh \frac{(2n-1)\pi L}{2H}}{\cosh \frac{(2n-1)\pi L}{2H}}$$

So then

$$\begin{aligned} X(x) &= C_3 \cosh \frac{(2n-1)\pi x}{2H} + C_4 \sinh \frac{(2n-1)\pi x}{2H} \\ &= -C_4 \frac{\sinh \frac{(2n-1)\pi L}{2H}}{\cosh \frac{(2n-1)\pi L}{2H}} \cosh \frac{(2n-1)\pi x}{2H} + C_4 \sinh \frac{(2n-1)\pi x}{2H} \\ &= -\frac{C_4}{\cosh \frac{(2n-1)\pi L}{2H}} \left[\sinh \frac{(2n-1)\pi L}{2H} \cosh \frac{(2n-1)\pi x}{2H} - \cosh \frac{(2n-1)\pi L}{2H} \sinh \frac{(2n-1)\pi x}{2H} \right] \\ &= -\frac{C_4}{\cosh \frac{(2n-1)\pi L}{2H}} \sinh \frac{(2n-1)\pi (L-x)}{2H} \quad \rightarrow \quad X_n(x) = \sinh \frac{(2n-1)\pi (L-x)}{2H}. \end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Y becomes

$$Y'' = 0.$$

Integrate both sides with respect to y.

$$Y' = C_5$$

Apply the first boundary condition to determine C_5.

$$Y'(0) = C_5 = 0$$

Consequently,

$$Y' = 0.$$

Integrate both sides with respect to y once more.

$$Y(y) = C_6$$

Apply the second boundary condition to determine C_6.

$$Y(H) = C_6 = 0$$

The trivial solution $Y(y) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Y becomes

$$Y'' = \beta^2 Y.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_7 \cosh \beta y + C_8 \sinh \beta y$$

Take the derivative of it.

$$Y'(y) = \beta(C_7 \sinh \beta y + C_8 \cosh \beta y)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} Y'(0) &= \beta(C_8) = 0 \\ Y(H) &= C_7 \cosh \beta H + C_8 \sinh \beta H = 0 \end{aligned}$$

The first equation implies that $C_8 = 0$, so the second one reduces to $C_7 \cosh \beta H = 0$. No nonzero value of β can satisfy this equation, so C_7 must be zero. The trivial solution $Y(y) = 0$ is obtained, which means there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{(2n-1)\pi(L-x)}{2H} \cos \frac{(2n-1)\pi y}{2H}}$$

Use the remaining inhomogeneous boundary condition $u(0,y) = g(y)$ to determine A_n.

$$u(0,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{(2n-1)\pi L}{2H} \cos \frac{(2n-1)\pi y}{2H} = g(y)$$

Multiply both sides by $\cos[(2m-1)\pi y/(2H)]$

$$\sum_{n=1}^{\infty} A_n \sinh \frac{(2n-1)\pi L}{2H} \cos \frac{(2n-1)\pi y}{2H} \cos \frac{(2m-1)\pi y}{2H} = g(y) \cos \frac{(2m-1)\pi y}{2H}$$

and then integrate both sides with respect to y from 0 to H.

$$\int_0^H \sum_{n=1}^{\infty} A_n \sinh \frac{(2n-1)\pi L}{2H} \cos \frac{(2n-1)\pi y}{2H} \cos \frac{(2m-1)\pi y}{2H}\, dy = \int_0^H g(y) \cos \frac{(2m-1)\pi y}{2H}\, dy$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} A_n \sinh \frac{(2n-1)\pi L}{2H} \int_0^H \cos \frac{(2n-1)\pi y}{2H} \cos \frac{(2m-1)\pi y}{2H}\, dy = \int_0^H g(y) \cos \frac{(2m-1)\pi y}{2H}\, dy$$

Because the cosine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ term.

$$A_n \sinh \frac{(2n-1)\pi L}{2H} \int_0^H \cos^2 \frac{(2n-1)\pi y}{2H}\, dy = \int_0^H g(y) \cos \frac{(2n-1)\pi y}{2H}\, dy$$

$$A_n \sinh \frac{(2n-1)\pi L}{2H} \left(\frac{H}{2}\right) = \int_0^H g(y) \cos \frac{(2n-1)\pi y}{2H}\, dy$$

So then

$$\boxed{A_n = \frac{2}{H \sinh \frac{(2n-1)\pi L}{2H}} \int_0^H g(y) \cos \frac{(2n-1)\pi y}{2H}\, dy.}$$

Part (e)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$
$$u(0, y) = 0$$
$$u(L, y) = 0$$
$$u(x, 0) - \frac{\partial u}{\partial y}(x, 0) = 0$$
$$u(x, H) = f(x)$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \to \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$u(0, y) = 0 \quad \to \quad X(0)Y(y) = 0 \quad \to \quad X(0) = 0$$
$$u(L, y) = 0 \quad \to \quad X(L)Y(y) = 0 \quad \to \quad X(L) = 0$$
$$u(x, 0) - \frac{\partial u}{\partial y}(x, 0) = 0 \quad \to \quad X(x)Y(0) - X(x)Y'(0) = 0 \quad \to \quad Y(0) - Y'(0) = 0$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for X first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$X'' = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(L) &= C_1 \cosh \alpha L + C_2 \sinh \alpha L = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh \alpha L = 0$. No nonzero value of α satisfies this equation, so C_2 must be zero. The trivial solution is obtained, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(L) &= C_3 L + C_4 = 0 \end{aligned}$$

The second equation reduces to $C_3 L = 0$, so $C_3 = 0$. The trivial solution $X(x) = 0$ is obtained, which means zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X(0) &= C_7 = 0 \\ X(L) &= C_7 \cos \beta L + C_8 \sin \beta L = 0 \end{aligned}$$

The second equation reduces to $C_8 \sin \beta L = 0$. To avoid getting the trivial solution, we insist that $C_8 \neq 0$. Then

$$\begin{aligned} \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \beta x + C_8 \sin \beta x \\ &= C_8 \sin \beta x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{L}. \end{aligned}$$

With this formula for λ, solve the ODE for Y now.

$$\frac{d^2Y}{dy^2} = \frac{n^2\pi^2}{L^2} Y$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_9 \cosh \frac{n\pi y}{L} + C_{10} \sinh \frac{n\pi y}{L}$$

Take a derivative of it.

$$Y'(y) = \frac{n\pi}{L}\left(C_9 \sinh \frac{n\pi y}{L} + C_{10} \cosh \frac{n\pi y}{L}\right)$$

Use the boundary condition to determine one of the constants.

$$Y(0) - Y'(0) = C_9 - \frac{n\pi}{L}(C_{10}) = 0 \quad \rightarrow \quad C_9 = \frac{n\pi}{L} C_{10}$$

So then

$$\begin{aligned} Y(y) &= C_9 \cosh \frac{n\pi y}{L} + C_{10} \sinh \frac{n\pi y}{L} \\ &= \frac{n\pi}{L} C_{10} \cosh \frac{n\pi y}{L} + C_{10} \sinh \frac{n\pi y}{L} \\ &= C_{10}\left(\sinh \frac{n\pi y}{L} + \frac{n\pi}{L} \cosh \frac{n\pi y}{L}\right) \quad \rightarrow \quad Y_n(y) = \sinh \frac{n\pi y}{L} + \frac{n\pi}{L} \cosh \frac{n\pi y}{L}. \end{aligned}$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}\left(\sinh \frac{n\pi y}{L} + \frac{n\pi}{L} \cosh \frac{n\pi y}{L}\right)}$$

Use the final inhomogeneous boundary condition $u(x, H) = f(x)$ to determine B_n.

$$u(x, H) = \sum_{n=1}^{\infty} B_n \left(\sinh \frac{n\pi H}{L} + \frac{n\pi}{L} \cosh \frac{n\pi H}{L}\right) \sin \frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin(m\pi x/L)$, where m is an integer,

$$\sum_{n=1}^{\infty} B_n \left(\sinh \frac{n\pi H}{L} + \frac{n\pi}{L} \cosh \frac{n\pi H}{L}\right) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = f(x) \sin \frac{m\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \left(\sinh \frac{n\pi H}{L} + \frac{n\pi}{L} \cosh \frac{n\pi H}{L}\right) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} B_n \left(\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right)\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$B_n \left(\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right)\int_0^L \sin^2\frac{n\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n \left(\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right)\left(\frac{L}{2}\right) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

So then

$$\boxed{B_n = \frac{2}{L\left(\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right)}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Part (f)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$
$$u(0, y) = f(y)$$
$$u(L, y) = 0$$
$$\frac{\partial u}{\partial y}(x, 0) = 0$$
$$\frac{\partial u}{\partial y}(x, H) = 0$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u(L, y) = 0 \quad &\rightarrow \quad X(L)Y(y) = 0 \quad &\rightarrow \quad X(L) = 0 \\ \frac{\partial u}{\partial y}(x, 0) = 0 \quad &\rightarrow \quad X(x)Y'(0) = 0 \quad &\rightarrow \quad Y'(0) = 0 \\ \frac{\partial u}{\partial y}(x, H) = 0 \quad &\rightarrow \quad X(x)Y'(H) = 0 \quad &\rightarrow \quad Y'(H) = 0 \end{aligned}$$

Separate variables in the PDE.

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2 X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2 Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called eigenvalues, and the solutions themselves are known as eigenfunctions. We will solve the ODE for Y first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Y becomes

$$Y'' = -\alpha^2 Y.$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_1 \cos \alpha y + C_2 \sin \alpha y$$

Take a derivative of it.

$$Y'(y) = \alpha(-C_1 \sin \alpha y + C_2 \cos \alpha y)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} Y'(0) &= \alpha(C_2) = 0 \\ Y'(H) &= \alpha(-C_1 \sin \alpha H + C_2 \cos \alpha H) = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second one reduces to $-C_1 \alpha \sin \alpha H = 0$. To avoid getting the trivial solution, we insist that $C_1 \neq 0$. Then

$$\begin{aligned} -\alpha \sin \alpha H &= 0 \\ \sin \alpha H &= 0 \\ \alpha H &= n\pi, \quad n = 1, 2, \ldots \\ \alpha_n &= \frac{n\pi}{H}. \end{aligned}$$

There are positive eigenvalues $\lambda = n^2\pi^2/H^2$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_1 \cos \alpha y + C_2 \sin \alpha y \\ &= C_1 \cos \alpha y \quad \rightarrow \quad Y_n(y) = \cos \frac{n\pi y}{H}. \end{aligned}$$

With this formula for λ, the ODE for X becomes

$$\frac{d^2X}{dx^2} = \frac{n^2\pi^2}{H^2} X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \frac{n\pi x}{H} + C_4 \sinh \frac{n\pi x}{H}$$

Apply the boundary condition to determine one of the constants.

$$X(L) = C_3 \cosh \frac{n\pi L}{H} + C_4 \sinh \frac{n\pi L}{H} = 0 \quad \rightarrow \quad C_3 = -C_4 \frac{\sinh \frac{n\pi L}{H}}{\cosh \frac{n\pi L}{H}}$$

So then

$$\begin{aligned} X(x) &= C_3 \cosh \frac{n\pi x}{H} + C_4 \sinh \frac{n\pi x}{H} \\ &= -C_4 \frac{\sinh \frac{n\pi L}{H}}{\cosh \frac{n\pi L}{H}} \cosh \frac{n\pi x}{H} + C_4 \sinh \frac{n\pi x}{H} \\ &= -\frac{C_4}{\cosh \frac{n\pi L}{H}} \left(\sinh \frac{n\pi L}{H} \cosh \frac{n\pi x}{H} - \cosh \frac{n\pi L}{H} \sinh \frac{n\pi x}{H} \right) \\ &= -\frac{C_4}{\cosh \frac{n\pi L}{H}} \sinh \frac{n\pi(L-x)}{H} \quad \rightarrow \quad X_n(x) = \sinh \frac{n\pi(L-x)}{H}. \end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Y becomes

$$Y'' = 0.$$

Integrate both sides with respect to y.

$$Y' = C_5$$

Apply the boundary conditions to determine C_5.

$$\begin{aligned} Y'(0) &= C_5 = 0 \\ Y'(H) &= C_5 = 0 \end{aligned}$$

Consequently,

$$Y' = 0.$$

Integrate both sides with respect to y once more.

$$Y(y) = C_6$$

Because $Y(y)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $Y_0(y) = 1$. Now solve the ODE for X with $\lambda = 0$.

$$X'' = 0$$

Integrate both sides with respect to x twice.

$$X(x) = C_7 x + C_8$$

Apply the boundary condition to determine one of the constants.

$$X(L) = C_7 L + C_8 = 0 \quad \rightarrow \quad C_8 = -C_7 L$$

Consequently,

$$\begin{aligned} X(x) &= C_7 x + C_8 \\ &= C_7 x - C_7 L \\ &= -C_7(L - x) \quad \rightarrow \quad X_n(x) = L - x. \end{aligned}$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Y becomes

$$Y'' = \beta^2 Y.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_7 \cosh \beta y + C_8 \sinh \beta y$$

Take the derivative of it.

$$Y'(y) = \beta(C_7 \sinh \beta y + C_8 \cosh \beta y)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} Y'(0) &= \beta(C_8) = 0 \\ Y'(H) &= \beta(C_7 \sinh \beta H + C_8 \cosh \beta H) = 0 \end{aligned}$$

The first equation implies that $C_8 = 0$, so the second one reduces to $C_7\beta\sinh\beta H = 0$. No nonzero value of β can satisfy this equation, so C_7 must be zero. The trivial solution $Y(y) = 0$ is obtained, which means there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x,y) = A_0(L-x)\cdot 1 + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi(L-x)}{H}\cos\frac{n\pi y}{H}}$$

Use the remaining inhomogeneous boundary condition $u(0,y) = f(y)$ to determine A_0 and A_n.

$$u(0,y) = A_0L + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L}{H}\cos\frac{n\pi y}{H} = f(y) \tag{2}$$

To find A_0, integrate both sides of equation (2) with respect to y from 0 to H.

$$\int_0^H \left(A_0L + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L}{H}\cos\frac{n\pi y}{H}\right) dy = \int_0^H f(y)\,dy$$

Split up the integral on the left and bring the constants in front.

$$A_0L\int_0^H dy + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L}{H}\underbrace{\int_0^H \cos\frac{n\pi y}{H}\,dy}_{=\,0} = \int_0^H f(y)\,dy$$

Evaluate the integrals.

$$A_0LH = \int_0^H f(y)\,dy$$

So then

$$\boxed{A_0 = \frac{1}{HL}\int_0^H f(y)\,dy.}$$

To find A_n, multiply both sides of equation (2) by $\cos(m\pi y/H)$, where m is an integer,

$$A_0L\cos\frac{m\pi y}{H} + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L}{H}\cos\frac{n\pi y}{H}\cos\frac{m\pi y}{H} = f(y)\cos\frac{m\pi y}{H}$$

and then integrate both sides with respect to y from 0 to H.

$$\int_0^H \left(A_0L\cos\frac{m\pi y}{H} + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L}{H}\cos\frac{n\pi y}{H}\cos\frac{m\pi y}{H}\right) dy = \int_0^H f(y)\cos\frac{m\pi y}{H}\,dy$$

Split up the integral on the left and bring the constants in front of them.

$$A_0L\underbrace{\int_0^H \cos\frac{m\pi y}{H}\,dy}_{=\,0} + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L}{H}\int_0^H \cos\frac{n\pi y}{H}\cos\frac{m\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{m\pi y}{H}\,dy$$

Because the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \sinh\frac{n\pi L}{H}\int_0^H \cos^2\frac{n\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{n\pi y}{H}\,dy$$

$$A_n \sinh \frac{n\pi L}{H} \left(\frac{H}{2} \right) = \int_0^H f(y) \cos \frac{n\pi y}{H}\, dy$$

So then

$$\boxed{A_n = \frac{2}{H \sinh \frac{n\pi L}{H}} \int_0^H f(y) \cos \frac{n\pi y}{H}\, dy.}$$

Part (g)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$

$$\frac{\partial u}{\partial x}(0, y) = 0$$

$$\frac{\partial u}{\partial x}(L, y) = 0$$

$$u(x, 0) = f(x) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$$

$$\frac{\partial u}{\partial y}(x, H) = 0$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial x}(0, y) = 0 \quad &\rightarrow \quad X'(0)Y(y) = 0 \quad &\rightarrow \quad X'(0) = 0 \\ \frac{\partial u}{\partial x}(L, y) = 0 \quad &\rightarrow \quad X'(L)Y(y) = 0 \quad &\rightarrow \quad X'(L) = 0 \\ \frac{\partial u}{\partial y}(x, H) = 0 \quad &\rightarrow \quad X(x)Y'(H) = 0 \quad &\rightarrow \quad Y'(H) = 0 \end{aligned}$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for X first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$X'' = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative of it.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X'(L) &= \alpha(C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0 \end{aligned}$$

The first equation implies that C_2, so the second one reduces to $C_1 \alpha \sinh \alpha L = 0$. No nonzero value of α satisfies this equation, so C_1 must be zero. The trivial solution is obtained, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_3$$

Apply the boundary conditions to determine C_3.

$$\begin{aligned} X'(0) &= C_3 = 0 \\ X'(L) &= C_3 = 0 \end{aligned}$$

Consequently,

$$X' = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Because $X(x)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $X_0(x) = 1$. With this value for λ, solve the ODE for Y.

$$Y'' = 0$$

Integrate both sides with respect to y.

$$Y' = C_5$$

Apply the boundary condition to determine one of the constants.

$$Y'(H) = C_5 = 0$$

So then

$$Y' = 0.$$

Integrate both sides with respect to y once more.

$$Y(y) = C_6$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Take a derivative of it.

$$X'(x) = \beta(-C_7 \sin \beta x + C_8 \cos \beta x)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X'(0) &= \beta(C_8) = 0 \\ X'(L) &= \beta(-C_7 \sin \beta L + C_8 \cos \beta L) = 0 \end{aligned}$$

The first equation implies that $C_8 = 0$, so the second one reduces to $-C_7 \beta \sin \beta L = 0$. To avoid getting the trivial solution, we insist that $C_7 \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta L &= 0 \\ \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \beta x + C_8 \sin \beta x \\ &= C_7 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}. \end{aligned}$$

With this formula for λ, solve the ODE for Y now.

$$\frac{d^2Y}{dy^2} = \frac{n^2\pi^2}{L^2} Y$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_9 \cosh \frac{n\pi y}{L} + C_{10} \sinh \frac{n\pi y}{L}$$

Take a derivative of it.

$$Y'(y) = \frac{n\pi}{L} \left(C_9 \sinh \frac{n\pi y}{L} + C_{10} \cosh \frac{n\pi y}{L} \right)$$

Use the boundary condition to determine one of the constants.

$$Y'(H) = \frac{n\pi}{L}\left(C_9 \sinh\frac{n\pi H}{L} + C_{10}\cosh\frac{n\pi H}{L}\right) = 0 \quad \rightarrow \quad C_{10} = -C_9 \frac{\sinh\frac{n\pi H}{L}}{\cosh\frac{n\pi H}{L}}$$

So then

$$\begin{aligned} Y(y) &= C_9 \cosh\frac{n\pi y}{L} + C_{10}\sinh\frac{n\pi y}{L} \\ &= C_9\cosh\frac{n\pi y}{L} - C_9\frac{\sinh\frac{n\pi H}{L}}{\cosh\frac{n\pi H}{L}}\sinh\frac{n\pi y}{L} \\ &= \frac{C_9}{\cosh\frac{n\pi H}{L}}\left(\cosh\frac{n\pi H}{L}\cosh\frac{n\pi y}{L} - \sinh\frac{n\pi H}{L}\sinh\frac{n\pi y}{L}\right) \\ &= \frac{C_9}{\cosh\frac{n\pi H}{L}}\cosh\frac{n\pi(H-y)}{L} \quad \rightarrow \quad Y_n(y) = \cosh\frac{n\pi(H-y)}{L}. \end{aligned}$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cosh\frac{n\pi(H-y)}{L}$$

Use the final inhomogeneous boundary condition $u(x,0) = f(x)$ to determine A_0 and A_n.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi H}{L}\cos\frac{n\pi x}{L} = f(x) \tag{3}$$

To find A_0, integrate both sides of equation (3) with respect to x from 0 to L.

$$\int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n\cosh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\right)dx = \int_0^L f(x)\,dx$$

Split up the integral on the left and bring the constants in front. Also, write out the integral on the right.

$$A_0\int_0^L dx + \sum_{n=1}^{\infty} A_n\cosh\frac{n\pi H}{L}\underbrace{\int_0^L \cos\frac{n\pi x}{L}\,dx}_{=\,0} = \int_0^{L/2}(1)\,dx + \int_{L/2}^L (0)\,dx$$

$$A_0 L = \frac{L}{2}$$

So then

$$A_0 = \frac{1}{2}.$$

To find A_n, multiply both sides of equation (3) by $\cos(m\pi x/L)$, where m is an integer,

$$A_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n\cosh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = f(x)\cos\frac{m\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \left(A_0 \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right) dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Split up the integral on the left and bring the constants in front. Also, write out the integral on the right.

$$\begin{aligned} A_0 \underbrace{\int_0^L \cos\frac{m\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi H}{L}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \\ = \int_0^{L/2}(1)\cos\frac{m\pi x}{L}\,dx + \int_{L/2}^{L}(0)\cos\frac{m\pi x}{L}\,dx \end{aligned}$$

Since the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \cosh\frac{n\pi H}{L}\int_0^L \cos^2\frac{n\pi x}{L}\,dx = \int_0^{L/2}\cos\frac{n\pi x}{L}\,dx$$

$$A_n \cosh\frac{n\pi H}{L}\left(\frac{L}{2}\right) = \frac{L}{n\pi}\sin\frac{n\pi}{2}$$

So then

$$A_n = \frac{2}{n\pi\cosh\frac{n\pi H}{L}}\sin\frac{n\pi}{2}$$

and

$$u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi\cosh\frac{n\pi H}{L}}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{L}\cosh\frac{n\pi(H-y)}{L}.$$

Notice that the summand is zero if n is even. The solution can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Make the substitution $n = 2p-1$ in the sum.

$$\begin{aligned} u(x,y) &= \frac{1}{2} + \sum_{2p-1=1}^{\infty}\frac{2}{(2p-1)\pi\cosh\frac{(2p-1)\pi H}{L}}\sin\frac{(2p-1)\pi}{2}\cos\frac{(2p-1)\pi x}{L}\cosh\frac{(2p-1)\pi(H-y)}{L} \\ &= \frac{1}{2} + \sum_{p=1}^{\infty}\frac{2}{(2p-1)\pi\cosh\frac{(2p-1)\pi H}{L}}[-(-1)^p]\cos\frac{(2p-1)\pi x}{L}\cosh\frac{(2p-1)\pi(H-y)}{L} \end{aligned}$$

Therefore,

$$u(x,y) = \frac{1}{2} - \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p-1)\cosh\frac{(2p-1)\pi H}{L}}\cos\frac{(2p-1)\pi x}{L}\cosh\frac{(2p-1)\pi(H-y)}{L}.$$

Part (h)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le H$$
$$u(0, y) = 0$$
$$u(L, y) = g(y)$$
$$u(x, 0) = 0$$
$$u(x, H) = 0$$

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\begin{array}{lllll} u(0, y) = 0 & \rightarrow & X(0)Y(y) = 0 & \rightarrow & X(0) = 0 \\ u(x, 0) = 0 & \rightarrow & X(x)Y(0) = 0 & \rightarrow & Y(0) = 0 \\ u(x, H) = 0 & \rightarrow & X(x)Y(H) = 0 & \rightarrow & Y(H) = 0 \end{array}$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for Y first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Y becomes

$$Y'' = -\alpha^2 Y.$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_1 \cos \alpha y + C_2 \sin \alpha y$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} Y(0) &= C_1 = 0 \\ Y(H) &= C_1 \cos \alpha H + C_2 \sin \alpha H = 0 \end{aligned}$$

The second equation reduces to $C_2 \sin \alpha H = 0$. To avoid getting the trivial solution, we insist that $C_2 \neq 0$. Then

$$\begin{aligned} \sin \alpha H &= 0 \\ \alpha H &= n\pi, \quad n = 1, 2, \ldots \\ \alpha_n &= \frac{n\pi}{H}. \end{aligned}$$

There are positive eigenvalues $\lambda = n^2\pi^2/H^2$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_1 \cos \alpha y + C_2 \sin \alpha y \\ &= C_2 \sin \alpha y \quad \to \quad Y_n(y) = \sin \frac{n\pi y}{H}. \end{aligned}$$

With this formula for λ, the ODE for X becomes

$$\frac{d^2X}{dx^2} = \frac{n^2\pi^2}{H^2} X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \frac{n\pi x}{H} + C_4 \sinh \frac{n\pi x}{H}$$

Apply the boundary condition to determine one of the constants.

$$X(0) = C_3 = 0$$

So then

$$X(x) = C_4 \sinh \frac{n\pi x}{H} \quad \to \quad X_n(x) = \sinh \frac{n\pi x}{H}.$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Y becomes

$$Y'' = 0.$$

Integrate both sides with respect to y twice.

$$Y(y) = C_5 y + C_6$$

Apply the boundary conditions to determine C_5 and C_6.

$$Y(0) = C_6 = 0$$
$$Y(H) = C_5 H + C_6 = 0$$

The second equation reduces to $C_5 H = 0$, which means $C_5 = 0$. The trivial solution $Y(y) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Y becomes

$$Y'' = \beta^2 Y.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_7 \cosh \beta y + C_8 \sinh \beta y$$

Apply the boundary conditions to determine C_7 and C_8.

$$Y(0) = C_7 = 0$$
$$Y(H) = C_7 \cosh \beta H + C_8 \sinh \beta H = 0$$

The second equation reduces to $C_8 \sinh \beta H = 0$. No nonzero value of β can satisfy this equation, so C_8 must be zero. The trivial solution $Y(y) = 0$ is obtained, which means there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}}$$

Use the remaining inhomogeneous boundary condition $u(L, y) = g(y)$ to determine B_n.

$$u(L,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} = g(y)$$

Multiply both sides by $\sin(m\pi y/H)$, where m is an integer,

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} \sin \frac{m\pi y}{H} = g(y) \sin \frac{m\pi y}{H}$$

and then integrate both sides with respect to y from 0 to H.

$$\int_0^H \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} \sin \frac{m\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{m\pi y}{H}\, dy$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \int_0^H \sin \frac{n\pi y}{H} \sin \frac{m\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{m\pi y}{H}\, dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$B_n \sinh \frac{n\pi L}{H} \int_0^H \sin^2 \frac{n\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$$

$$B_n \sinh \frac{n\pi L}{H} \left(\frac{H}{2}\right) = \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$$

So then

$$\boxed{B_n = \frac{2}{H \sinh \frac{n\pi L}{H}} \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy.}$$