Exercise 2.5.4

For Laplace's equation inside a circular disk ($r \leq a$), using (2.5.45) and (2.5.47), show that

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\left[-\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \bar{\theta})\right] d\bar{\theta}.$$

Using $\cos z = \operatorname{Re}[e^{iz}]$, sum the resulting geometric series to obtain Poisson's integral formula.

Solution

Here we will solve the Laplace equation inside a disk.

$$\begin{aligned}
&\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad r \leq a,\ 0 \leq \theta \leq 2\pi \\
&u(a,\theta) = f(\theta) \\
&u(r,\theta) \text{ bounded as } r \to 0
\end{aligned}$$

Because the boundary condition of the Laplace equation is prescribed on a circle, the method of separation of variables can be applied. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and substitute it into the PDE.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \to \quad \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}[R(r)\Theta(\theta)]\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[R(r)\Theta(\theta)] = 0$$

Proceed to separate variables.

$$\frac{\Theta}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

Bring the second term to the right side. (The final answer will be the same regardless which side the minus sign is on.)

$$\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{function of } r} = \underbrace{-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}}_{\text{function of } \theta}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of applying the method of separation of variables, the Laplace equation has been reduced to two ODEs—one in r and one in θ.

$$\left.\begin{aligned}
\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= \lambda \\
-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} &= \lambda
\end{aligned}\right\}$$

Periodic boundary conditions are assumed for Θ, since the solution and its slope in the θ-direction are expected to be the same at $\theta = 0$ and $\theta = 2\pi$.

$$\Theta(0) = \Theta(2\pi)$$
$$\frac{d\Theta}{d\theta}(0) = \frac{d\Theta}{d\theta}(2\pi)$$

Values of λ for which nontrivial solutions of the preceding equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Θ becomes

$$\frac{d^2\Theta}{d\theta^2} = -\alpha^2\Theta.$$

The general solution is written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\alpha\theta + C_2 \sin\alpha\theta$$

Take the derivative of it.

$$\Theta'(\theta) = \alpha(-C_1 \sin\alpha\theta + C_2 \cos\alpha\theta)$$

Apply the boundary conditions to obtain a system of equations involving C_1 and C_2.

$$\Theta(0) = C_1 = C_1 \cos 2\pi\alpha + C_2 \sin 2\pi\alpha = \Theta(2\pi)$$
$$\Theta'(0) = \alpha(C_2) = \alpha(-C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha) = \Theta'(2\pi)$$

$$\begin{cases} C_1 = C_1 \cos 2\pi\alpha + C_2 \sin 2\pi\alpha \\ C_2 = -C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha \end{cases}$$

$$\begin{cases} C_1(1 - \cos 2\pi\alpha) = C_2 \sin 2\pi\alpha \\ C_2(1 - \cos 2\pi\alpha) = -C_1 \sin 2\pi\alpha \end{cases}$$

These equations are satisfied if $\alpha = n$, where $n = 1, 2, \ldots$. The positive eigenvalues are thus $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos\alpha\theta + C_2 \sin\alpha\theta \quad \rightarrow \quad \Theta_n(\theta) = C_1 \cos n\theta + C_2 \sin n\theta.$$

With this formula for λ, the ODE for R becomes

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0.$$

This ODE is equidimensional, so the general solution is of the form $R(r) = r^k$. Find its derivatives

$$R(r) = r^k \quad \rightarrow \quad \frac{dR}{dr} = kr^{k-1} \quad \rightarrow \quad \frac{d^2R}{dr^2} = k(k-1)r^{k-2}$$

and substitute them into the equation.

$$r^2k(k-1)r^{k-2} + rkr^{k-1} - n^2r^k = 0$$

$$k(k-1)r^k + kr^k - n^2 r^k = 0$$

Divide both sides by r^k.

$$k(k-1) + k - n^2 = 0$$

$$k^2 - n^2 = 0$$

$$k = \pm n$$

Consequently,

$$R(r) = C_3 r^{-n} + C_4 r^n.$$

Since u remains finite as $r \to 0$, we require that $C_3 = 0$.

$$R(r) = C_4 r^n$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Θ becomes

$$\Theta'' = 0.$$

Integrate both sides with respect to θ.

$$\Theta' = C_5$$

Integrate both sides with respect to θ once more.

$$\Theta(\theta) = C_5\theta + C_6$$

Apply the boundary conditions to obtain a system of equations involving C_5 and C_6.

$$\begin{aligned}\Theta(0) = C_6 &= 2\pi C_5 + C_6 = \Theta(2\pi)\\ \Theta'(0) = C_5 &= C_5 = \Theta'(2\pi)\end{aligned}$$

The first equation implies that $C_5 = 0$ and C_6 is arbitrary, and the second equation gives no information.

$$\Theta(\theta) = C_6$$

Since $\Theta(\theta)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $\Theta_0(\theta) = 1$. Now solve the ODE for R with $\lambda = 0$.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = C_7$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{C_7}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = C_7 \ln r + C_8$$

For u to remain finite as $r \to 0$, we require that $C_7 = 0$.

$$R(r) = C_8$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Θ becomes

$$\frac{d^2\Theta}{d\theta^2} = \beta^2\Theta.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_9 \cosh \beta\theta + C_{10} \sinh \beta\theta$$

Take the derivative of it.

$$\Theta'(\theta) = \beta(C_9 \sinh \beta\theta + C_{10} \cosh \beta\theta)$$

Apply the boundary conditions to obtain a system of equations involving C_9 and C_{10}.

$$\begin{aligned} \Theta(0) = C_9 &= C_9 \cosh 2\pi\beta + C_{10} \sinh 2\pi\beta = \Theta(2\pi) \\ \Theta'(0) = \beta(C_{10}) &= \beta(C_9 \sinh 2\pi\beta + C_{10} \cosh 2\pi\beta) = \Theta'(2\pi) \end{aligned}$$

$$\begin{cases} C_9 = C_9 \cosh 2\pi\beta + C_{10} \sinh 2\pi\beta \\ C_{10} = C_9 \sinh 2\pi\beta + C_{10} \cosh 2\pi\beta \end{cases}$$

$$\begin{cases} C_9(1 - \cosh 2\pi\beta) = C_{10} \sinh 2\pi\beta \\ C_{10}(1 - \cosh 2\pi\beta) = C_9 \sinh 2\pi\beta \end{cases}$$

No nonzero value of β satisfies these equations, so $C_9 = 0$ and $C_{10} = 0$. The trivial solution $\Theta(\theta) = 0$ is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $R(r)\Theta(\theta)$ over all the eigenvalues.

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{1}$$

Apply the boundary condition $u(a, \theta) = f(\theta)$ to determine the coefficients, A_0, A_n, and B_n.

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta) = f(\theta) \tag{2}$$

To find A_0, integrate both sides of equation (2) with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left[A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta) \right] d\theta = \int_0^{2\pi} f(\theta)\, d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} a^n \left(A_n \underbrace{\int_0^{2\pi} \cos n\theta\, d\theta}_{=\,0} + B_n \underbrace{\int_0^{2\pi} \sin n\theta\, d\theta}_{=\,0} \right) = \int_0^{2\pi} f(\theta)\, d\theta$$

$$A_0(2\pi) = \int_0^{2\pi} f(\theta)\, d\theta$$

So then

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta. \tag{3}$$

To find A_n, multiply both sides of equation (2) by $\cos m\theta$, where m is an integer,

$$A_0 \cos m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) = f(\theta) \cos m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left[A_0 \cos m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) \right] d\theta = \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \underbrace{\int_0^{2\pi} \cos m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} a^n \left(A_n \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta + B_n \underbrace{\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta}_{=0} \right)$$
$$= \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta$$

Because the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$a^n A_n \int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$a^n A_n(\pi) = \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

So then

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta. \tag{4}$$

To find B_n, multiply both sides of equation (2) by $\sin m\theta$, where m is an integer,

$$A_0 \sin m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) = f(\theta) \sin m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left[A_0 \sin m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) \right] d\theta = \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \underbrace{\int_0^{2\pi} \sin m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} a^n \left(A_n \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta}_{=0} + B_n \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta \right)$$
$$= \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta$$

Because the sine functions are orthogonal, the third integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$a^n B_n \int_0^{2\pi} \sin^2 n\theta \, d\theta = \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

$$a^n B_n(\pi) = \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

So then

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta. \tag{5}$$

Now substitute equations (3), (4), and (5) into equation (1).

$$\begin{aligned}
u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta})\, d\bar{\theta} + \sum_{n=1}^{\infty} r^n \left[\left(\frac{1}{\pi a^n} \int_0^{2\pi} f(\bar{\theta}) \cos n\bar{\theta} \, d\bar{\theta} \right) \cos n\theta + \left(\frac{1}{\pi a^n} \int_0^{2\pi} f(\bar{\theta}) \sin n\bar{\theta} \, d\bar{\theta} \right) \sin n\theta \right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta})\, d\bar{\theta} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^n}{a^n} \left[\int_0^{2\pi} f(\bar{\theta}) \cos n\theta \cos n\bar{\theta} \, d\bar{\theta} + \int_0^{2\pi} f(\bar{\theta}) \sin n\theta \sin n\bar{\theta} \, d\bar{\theta} \right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta})\, d\bar{\theta} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \int_0^{2\pi} f(\bar{\theta}) (\cos n\theta \cos n\bar{\theta} + \sin n\theta \sin n\bar{\theta}) \, d\bar{\theta} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta})\, d\bar{\theta} + \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \cos(n\theta - n\bar{\theta}) \, d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \operatorname{Re} e^{in(\theta - \bar{\theta})} \right] d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[\frac{1}{2} + \operatorname{Re} \sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta - \bar{\theta})} \right)^n \right] d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \operatorname{Re} \sum_{n=0}^{\infty} \left(\frac{r}{a} e^{i(\theta - \bar{\theta})} \right)^n \right] d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \operatorname{Re} \frac{1}{1 - \frac{r}{a} e^{i(\theta - \bar{\theta})}} \right] d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \operatorname{Re} \frac{1}{1 - \frac{r}{a}\cos(\theta - \bar{\theta}) - i\frac{r}{a}\sin(\theta - \bar{\theta})} \cdot \frac{1 - \frac{r}{a}\cos(\theta - \bar{\theta}) + i\frac{r}{a}\sin(\theta - \bar{\theta})}{1 - \frac{r}{a}\cos(\theta - \bar{\theta}) + i\frac{r}{a}\sin(\theta - \bar{\theta})} \right] d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \operatorname{Re} \frac{1 - \frac{r}{a}\cos(\theta - \bar{\theta}) + i\frac{r}{a}\sin(\theta - \bar{\theta})}{\left[1 - \frac{r}{a}\cos(\theta - \bar{\theta})\right]^2 + \frac{r^2}{a^2}\sin^2(\theta - \bar{\theta})} \right] d\bar{\theta} \\
&= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \operatorname{Re} \frac{1 - \frac{r}{a}\cos(\theta - \bar{\theta}) + i\frac{r}{a}\sin(\theta - \bar{\theta})}{1 - \frac{2r}{a}\cos(\theta - \bar{\theta}) + \frac{r^2}{a^2}} \right] d\bar{\theta}
\end{aligned}$$

Continue simplifying the right side.

$$\begin{aligned}
u(r,\theta) &= \frac{1}{\pi}\int_0^{2\pi} f(\bar{\theta})\left[-\frac{1}{2} + \frac{1-\frac{r}{a}\cos(\theta-\bar{\theta})}{1-\frac{2r}{a}\cos(\theta-\bar{\theta})+\frac{r^2}{a^2}}\right]d\bar{\theta} \\
&= \frac{1}{\pi}\int_0^{2\pi} f(\bar{\theta})\frac{-1+\frac{2r}{a}\cos(\theta-\bar{\theta})-\frac{r^2}{a^2}+2-\frac{2r}{a}\cos(\theta-\bar{\theta})}{2\left[1-\frac{2r}{a}\cos(\theta-\bar{\theta})+\frac{r^2}{a^2}\right]}\,d\bar{\theta} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\bar{\theta})\frac{1-\frac{r^2}{a^2}}{1-\frac{2r}{a}\cos(\theta-\bar{\theta})+\frac{r^2}{a^2}}\,d\bar{\theta} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\bar{\theta})\frac{a^2-r^2}{a^2-2ar\cos(\theta-\bar{\theta})+r^2}\,d\bar{\theta}
\end{aligned}$$

Therefore,

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\bar{\theta})}{a^2-2ar\cos(\theta-\bar{\theta})+r^2}\,d\bar{\theta}.$$

Note that $f(\theta)$ was defined for $0 \le \theta \le 2\pi$. If one defines it for $-\pi \le \theta \le \pi$, then the limits of integration will go from $-\pi$ to π instead.