Exercise 2.5.13

Prove that the temperature satisfying Laplace's equation cannot attain its minimum in the interior.

Solution

Assume that the steady-state temperature in some three-dimensional domain D satisfies the Laplace equation.

$$\nabla^2 u = 0$$

Suppose that the minimum value of u is obtained somewhere inside D. Multiply both sides by -1.

$$-\nabla^2 u = 0$$

Bring the minus sign inside the Laplacian operator.

$$\nabla^2(-u) = 0$$

Let $v = -u$.

$$\nabla^2 v = 0$$

v also satisfies the Laplace equation. The maximum value of v is at the same location as the minimum of u, and its value is the negative of the minimum value of u. But according to the maximum principle for the Laplace equation, this maximum must be on the boundary of D, not inside it. This contradicts the initial assumption about the minimum of u. Therefore, the minimum of u lies somewhere on the boundary of D.