Exercise 2.5.14

Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$, is *not* well-posed. [*Hint:* Show that if the data are changed an arbitrarily small amount, for example,

$$f(x) \to f(x) + \frac{1}{n}\sin\frac{n\pi x}{L}$$

for large n, then the solution $u(x,t)$ changes by a large amount.]

Solution

The backward heat equation and the boundary conditions are linear and homogeneous, so the method of separation of variables will be applied to solve it. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2} \quad \to \quad \frac{\partial}{\partial t}[X(x)T(t)] = -k\frac{\partial^2}{\partial x^2}[X(x)T(t)] \quad \to \quad XT' = -kX''T$$

and the boundary conditions.

$$\begin{aligned} u(0,t) = 0 \quad &\to \quad X(0)T(t) = 0 \quad &\to \quad X(0) = 0 \\ u(L,t) = 0 \quad &\to \quad X(L)T(t) = 0 \quad &\to \quad X(L) = 0 \end{aligned}$$

Divide both sides of the PDE by $kX(x)T(t)$ to separate variables.

$$\frac{T'}{kT} = -\frac{X''}{X}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T'}{kT} = -\frac{X''}{X} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned} \frac{T'}{kT} &= \lambda \\ -\frac{X''}{X} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Solve the ODE for X.

$$X'' = -\lambda X$$

Check for positive eigenvalues: $\lambda = \mu^2$.

$$X'' = -\mu^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(L) &= C_1 \cos \mu L + C_2 \sin \mu L = 0 \end{aligned}$$

This first equation makes the second one reduce to $C_2 \sin \mu L = 0$. In order to avoid getting the trivial solution, we insist that $C_2 \neq 0$.

$$\begin{aligned} \sin \mu L &= 0 \\ \mu L &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= \frac{n\pi}{L} \end{aligned}$$

There are positive eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$, and the eigenfunctions associated with them are

$$X(x) = C_2 \sin \mu x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{L}.$$

Note that only positive values of n are considered because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values of λ. With $\lambda = \frac{n^2\pi^2}{L^2}$, solve the ODE for T now.

$$\frac{T'}{kT} = \frac{n^2\pi^2}{L^2}$$

The general solution is an exponential function.

$$T(t) = B \exp\left(k \frac{n^2\pi^2}{L^2} t\right)$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$X'' = 0$$

The general solution is a straight line.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(L) &= C_3 L + C_4 = 0 \end{aligned}$$

The first equation makes the second one reduce to $C_3 L = 0$, which means $C_3 = 0$.

$$X(x) = 0$$

The trivial solution is obtained, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$X'' = \gamma^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(L) &= C_5 \cosh \gamma L + C_6 \sinh \gamma L = 0 \end{aligned}$$

The first equation makes the second one reduce to $C_6 \sinh \gamma L = 0$. No nonzero value of γ can satisfy this equation, so $C_6 = 0$.

$$X(x) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions $u = X_n(x)T_n(t)$ over all the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(k\frac{n^2\pi^2}{L^2}t\right) \sin \frac{n\pi x}{L}}$$

Use the final boundary condition to determine the constants B_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin \frac{p\pi x}{L}$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} = f(x) \sin \frac{p\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \left(\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\right) dx = \int_0^L f(x) \sin \frac{p\pi x}{L}\, dx$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx = \int_0^L f(x) \sin \frac{p\pi x}{L}\, dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq p$. Only if $n = p$ does this integral yield a nonzero result.

$$B_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Evaluate the integral.

$$B_n \left(\frac{L}{2}\right) = \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Therefore,

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Even though a solution was obtained to the backward heat equation with the method of separation of variables, there is a problem: It lacks stability. After a very long time τ, the exponential function makes the solution u astronomical in size. The boundary condition $u(0,t) = 0$ remains satisfied for all time, so $u(0,\tau) = 0$. Moving just a little bit to the right, for example, $u(0.01,\tau)$ results in a sudden jump in the value of u. In other words, a small change in (x,t) does not result in a similarly small change in u. The problem is not well-posed.