Exercise 2.5.28

Solve Laplace's equation inside a rectangle:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions

$$\begin{aligned} u(0, y) &= g(y) \qquad & u(x, 0) &= 0 \\ u(L, y) &= 0 \qquad & u(x, H) &= 0. \end{aligned}$$

Solution

Because the Laplace equation is linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution $u(x, y) = X(x)Y(y)$ and plug it into the PDE

$$\frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0 \quad \to \quad X''Y + XY'' = 0 \quad \to \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u(L, y) = 0 \quad &\to \quad X(L)Y(y) = 0 \quad &\to \quad X(L) = 0 \\ u(x, 0) = 0 \quad &\to \quad X(x)Y(0) = 0 \quad &\to \quad Y(0) = 0 \\ u(x, H) = 0 \quad &\to \quad X(x)Y(H) = 0 \quad &\to \quad Y(H) = 0 \end{aligned}$$

Separate variables in the PDE.

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned} \frac{X''}{X} &= \lambda \\ -\frac{Y''}{Y} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter what side the minus sign is on as long as all eigenvalues are considered. Solve the ODE for Y.

$$Y'' = -\lambda Y$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y$$

Apply the two boundary conditions to determine C_1 and C_2.

$$\begin{aligned} Y(0) &= C_1 = 0 \\ Y(H) &= C_1 \cos \mu H + C_2 \sin \mu H = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sin \mu H = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$.

$$\begin{aligned} \sin \mu H &= 0 \\ \mu H &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= \frac{n\pi}{H} \end{aligned}$$

There are positive eigenvalues $\lambda = \left(\frac{n\pi}{H}\right)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_2 \sin \mu y \quad \to \quad Y_n(y) = \sin \frac{n\pi y}{H}.$$

Note that n is taken over the positive integers only because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values for λ. Using $\lambda = \frac{n^2\pi^2}{H^2}$, solve the ODE for X now.

$$X'' = \frac{n^2\pi^2}{H^2} X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = A \cosh \frac{n\pi x}{H} + B \sinh \frac{n\pi x}{H}$$

Apply the boundary condition $X(L) = 0$ to determine one of the constants.

$$X(L) = A \cosh \frac{n\pi L}{H} + B \sinh \frac{n\pi L}{H} = 0 \quad \to \quad A = -\frac{\sinh \frac{n\pi L}{H}}{\cosh \frac{n\pi L}{H}} B$$

As a result, the X-eigenfunction becomes

$$\begin{aligned} X(x) &= -\frac{\sinh \frac{n\pi L}{H}}{\cosh \frac{n\pi L}{H}} B \cosh \frac{n\pi x}{H} + B \sinh \frac{n\pi x}{H} \\ &= -\frac{B}{\cosh \frac{n\pi L}{H}} \left(\sinh \frac{n\pi L}{H} \cosh \frac{n\pi x}{H} - \sinh \frac{n\pi x}{H} \cosh \frac{n\pi L}{H} \right) \\ &= -\frac{B}{\cosh \frac{n\pi L}{H}} \sinh \left(\frac{n\pi L}{H} - \frac{n\pi x}{H} \right) \quad \to \quad X_n(x) = \sinh \frac{n\pi(L - x)}{H}. \end{aligned}$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$Y'' = 0$$

The general solution is a straight line.

$$Y(y) = C_3 y + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} Y(0) &= C_4 = 0 \\ Y(H) &= C_3 H + C_4 = 0 \end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 H = 0$, which means $C_3 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_5 \cosh \gamma y + C_6 \sinh \gamma y$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} Y(0) &= C_5 = 0 \\ Y(H) &= C_5 \cosh \gamma H + C_6 \sinh \gamma H = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sinh \gamma H = 0$. No nonzero value of γ can satisfy this equation, so $C_6 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions $u = X_n(x)Y_n(y)$ over all the eigenvalues.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi(L-x)}{H} \sin \frac{n\pi y}{H}}$$

Apply the final boundary condition to determine the coefficients B_n.

$$u(0, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} = g(y)$$

Multiply both sides by $\sin \frac{p\pi y}{H}$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H} = g(y) \sin \frac{p\pi y}{H}$$

Integrate both sides with respect to y from 0 to H.

$$\int_0^H \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{p\pi y}{H}\, dy$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi L}{H} \int_0^H \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{p\pi y}{H}\, dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$B_n \sinh \frac{n\pi L}{H} \int_0^H \sin^2 \frac{n\pi y}{H}\, dy = \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$$

Evaluate the integral.

$$B_n \sinh \frac{n\pi L}{H} \left(\frac{H}{2}\right) = \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$$

Therefore,

$$\boxed{B_n = \frac{2}{H \sinh \frac{n\pi L}{H}} \int_0^H g(y) \sin \frac{n\pi y}{H}\, dy.}$$