Exercise 2.5.5

Solve Laplace's equation inside the quarter-circle of radius 1 ($0 \leq \theta \leq \pi/2$, $0 \leq r \leq 1$) subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

$$\begin{array}{llll}
\textbf{(a)} & \dfrac{\partial u}{\partial \theta}(r,0) = 0, & u\left(r,\dfrac{\pi}{2}\right) = 0, & u(1,\theta) = f(\theta) \\
\textbf{(b)} & \dfrac{\partial u}{\partial \theta}(r,0) = 0, & \dfrac{\partial u}{\partial \theta}\left(r,\dfrac{\pi}{2}\right) = 0, & u(1,\theta) = f(\theta) \\
\textbf{(c)} & u(r,0) = 0, & u\left(r,\dfrac{\pi}{2}\right) = 0, & \dfrac{\partial u}{\partial r}(1,\theta) = f(\theta) \\
\textbf{(d)} & \dfrac{\partial u}{\partial \theta}(r,0) = 0, & \dfrac{\partial u}{\partial \theta}\left(r,\dfrac{\pi}{2}\right) = 0, & \dfrac{\partial u}{\partial r}(1,\theta) = g(\theta)
\end{array}$$

Show that the solution [part (d)] exists only if $\int_0^{\pi/2} g(\theta)\,d\theta = 0$. Explain this condition physically.

Solution

Because the Laplace equation and all but one of its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and plug it into the PDE.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\Theta(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R(r)\Theta(\theta) = 0$$

$$\frac{\Theta(\theta)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$ in order to separate variables.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned} \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= \lambda \\ -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter whether the minus sign is grouped with r or θ as long as all eigenvalues are taken into account.

Part (a)

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial \theta}(r,0) = 0 \quad &\rightarrow \quad R(r)\Theta'(0) = 0 \quad &\rightarrow \quad \Theta'(0) = 0 \\ u\left(r,\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad R(r)\Theta\left(\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad \Theta\left(\frac{\pi}{2}\right) = 0 \end{aligned}$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin\mu\theta + C_2 \cos\mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta'(0) &= \mu(C_2) = 0 \\ \Theta\left(\frac{\pi}{2}\right) &= C_1 \cos\mu\frac{\pi}{2} + C_2 \sin\mu\frac{\pi}{2} = 0 \end{aligned}$$

This first equation gives $C_2 = 0$, which makes the second equation reduce to $C_1 \cos\mu\frac{\pi}{2} = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\begin{aligned} \cos\mu\frac{\pi}{2} &= 0 \\ \mu\frac{\pi}{2} &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots \\ \mu &= 2n-1 \end{aligned}$$

There are positive eigenvalues $\lambda = (2n-1)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \cos[(2n-1)\theta].$$

Using $\lambda = (2n-1)^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = (2n-1)^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = (2n-1)^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - (2n-1)^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2m(m-1)r^{m-2} + rmr^{m-1} - (2n-1)^2r^m = 0$$

$$m(m-1)r^m + mr^m - (2n-1)^2r^m = 0$$

$$m(m-1) + m - (2n-1)^2 = 0$$

$$m^2 - (2n-1)^2 = 0$$

$$[m + (2n-1)][m - (2n-1)] = 0$$

$$m = \{-(2n-1), 2n-1\}$$

Two solutions to the ODE are $R = r^{-(2n-1)}$ and $R = r^{2n-1}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-(2n-1)} + Br^{2n-1}$$

Now check to see if zero is an eigenvalue.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta'(0) = C_3 = 0$$

$$\Theta\left(\frac{\pi}{2}\right) = C_3\frac{\pi}{2} + C_4 = 0$$

This first equation makes the second equation reduce to $C_4 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh \gamma\theta + C_6 \sinh \gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh \gamma\theta + C_6 \cosh \gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\begin{aligned} \Theta'(0) &= \gamma(C_6) = 0 \\ \Theta\left(\frac{\pi}{2}\right) &= C_5 \cosh \gamma\frac{\pi}{2} + C_6 \sinh \gamma\frac{\pi}{2} = 0 \end{aligned}$$

This first equation gives $C_6 = 0$, which makes the second equation reduce to $C_5 \cosh \gamma\frac{\pi}{2} = 0$. No value of γ can satisfy this equation, so $C_5 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r, \theta) = \sum_{n=1}^{\infty} \left[A_n r^{-(2n-1)} + B_n r^{2n-1}\right] \cos[(2n-1)\theta]$$

For the solution to remain finite as $r \to 0$, set $A_n = 0$.

$$\boxed{u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{2n-1} \cos[(2n-1)\theta]}$$

Use the boundary condition at $r = 1$ to determine the remaining constants B_n.

$$u(1, \theta) = \sum_{n=1}^{\infty} B_n \cos[(2n-1)\theta] = f(\theta)$$

Multiply both sides by $\cos[(2p-1)\theta]$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n \cos[(2n-1)\theta] \cos[(2p-1)\theta] = f(\theta) \cos[(2p-1)\theta]$$

Integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2} \sum_{n=1}^{\infty} B_n \cos[(2n-1)\theta] \cos[(2p-1)\theta]\, d\theta = \int_0^{\pi/2} f(\theta) \cos[(2p-1)\theta]\, d\theta$$

Bring the constants in front of the integral.

$$\sum_{n=1}^{\infty} B_n \int_0^{\pi/2} \cos[(2n-1)\theta] \cos[(2p-1)\theta]\, d\theta = \int_0^{\pi/2} f(\theta) \cos[(2p-1)\theta]\, d\theta$$

Because the cosine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n \int_0^{\pi/2} \cos^2[(2n-1)\theta]\, d\theta = \int_0^{\pi/2} f(\theta) \cos[(2n-1)\theta]\, d\theta$$

$$B_n \left(\frac{\pi}{4}\right) = \int_0^{\pi/2} f(\theta) \cos[(2n-1)\theta]\, d\theta$$

Therefore,

$$\boxed{B_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \cos[(2n-1)\theta]\, d\theta.}$$

Part (b)

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial \theta}(r,0) = 0 \quad &\rightarrow \quad R(r)\Theta'(0) = 0 \quad \rightarrow \quad \Theta'(0) = 0 \\ \frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad R(r)\Theta'\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad \Theta'\left(\frac{\pi}{2}\right) = 0 \end{aligned}$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin\mu\theta + C_2 \cos\mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta'(0) &= \mu(C_2) = 0 \\ \Theta'\left(\frac{\pi}{2}\right) &= \mu\left(-C_1 \sin\mu\frac{\pi}{2} + C_2 \cos\mu\frac{\pi}{2}\right) = 0 \end{aligned}$$

This first equation gives $C_2 = 0$, which makes the second equation reduce to $-C_1\mu\sin\mu\frac{\pi}{2} = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\begin{aligned} \sin\mu\frac{\pi}{2} &= 0 \\ \mu\frac{\pi}{2} &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= 2n \end{aligned}$$

There are positive eigenvalues $\lambda = (2n)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \cos 2n\theta.$$

Using $\lambda = 4n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 4n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = 4n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - 4n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - 4n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - 4n^2 r^m = 0$$

$$m(m-1) + m - 4n^2 = 0$$

$$m^2 - 4n^2 = 0$$

$$(m+2n)(m-2n) = 0$$

$$m = \{-2n, 2n\}$$

Two solutions to the ODE are $R = r^{-2n}$ and $R = r^{2n}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-2n} + Br^{2n}$$

Now check to see if zero is an eigenvalue.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta'(0) = C_3 = 0$$

$$\Theta'\left(\frac{\pi}{2}\right) = C_3 = 0$$

C_4 remains arbitrary.

$$\Theta(\theta) = C_4$$

The trivial solution is not obtained, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = D_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{D_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = D_1 \ln r + D_2$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh \gamma\theta + C_6 \sinh \gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh \gamma\theta + C_6 \cosh \gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\begin{aligned} \Theta'(0) &= \gamma(C_6) = 0 \\ \Theta'\left(\frac{\pi}{2}\right) &= \gamma\left(C_5 \sinh \gamma\frac{\pi}{2} + C_6 \cosh \gamma\frac{\pi}{2}\right) = 0 \end{aligned}$$

This first equation gives $C_6 = 0$, which makes the second equation reduce to $C_5\gamma \sinh \gamma\frac{\pi}{2} = 0$. No nonzero value of γ can satisfy this equation, so $C_5 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r, \theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty}(A_n r^{-2n} + B_n r^{2n}) \cos 2n\theta$$

For the solution to remain finite as $r \to 0$, set $A_0 = 0$ and $A_n = 0$.

$$\boxed{u(r, \theta) = B_0 + \sum_{n=1}^{\infty} B_n r^{2n} \cos 2n\theta}$$

Use the boundary condition at $r = 1$ to determine the remaining constants, B_0 and B_n.

$$u(1, \theta) = B_0 + \sum_{n=1}^{\infty} B_n \cos 2n\theta = f(\theta) \tag{1}$$

To find B_0, integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2} \left(B_0 + \sum_{n=1}^{\infty} B_n \cos 2n\theta\right) d\theta = \int_0^{\pi/2} f(\theta)\, d\theta$$

Split up the integral on the left side and bring the constants in front.

$$B_0 \underbrace{\int_0^{\pi/2} d\theta}_{=\pi/2} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{\pi/2} \cos 2n\theta \, d\theta}_{=0} = \int_0^{\pi/2} f(\theta) \, d\theta$$

$$B_0 \left(\frac{\pi}{2}\right) = \int_0^{\pi/2} f(\theta) \, d\theta$$

Therefore,

$$\boxed{B_0 = \frac{2}{\pi} \int_0^{\pi/2} f(\theta) \, d\theta.}$$

To get B_n, multiply both sides of equation (1) by $\cos 2p\theta$, where p is an integer.

$$B_0 \cos 2p\theta + \sum_{n=1}^{\infty} B_n \cos 2n\theta \cos 2p\theta = f(\theta) \cos 2p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2} \left(B_0 \cos 2p\theta + \sum_{n=1}^{\infty} B_n \cos 2n\theta \cos 2p\theta \right) d\theta = \int_0^{\pi/2} f(\theta) \cos 2p\theta \, d\theta$$

Split up the integral and bring the constants in front.

$$B_0 \underbrace{\int_0^{\pi/2} \cos 2p\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} B_n \int_0^{\pi/2} \cos 2n\theta \cos 2p\theta \, d\theta = \int_0^{\pi/2} f(\theta) \cos 2p\theta \, d\theta$$

Because the cosine functions are orthogonal, this second integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n \int_0^{\pi/2} \cos^2 2n\theta \, d\theta = \int_0^{\pi/2} f(\theta) \cos 2n\theta \, d\theta$$

$$B_n \left(\frac{\pi}{4}\right) = \int_0^{\pi/2} f(\theta) \cos 2n\theta \, d\theta$$

Therefore,

$$\boxed{B_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \cos 2n\theta \, d\theta.}$$

Part (c)

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$\begin{aligned} u(r,0) &= 0 & \rightarrow \quad R(r)\Theta(0) &= 0 & \rightarrow \quad \Theta(0) &= 0 \\ u\left(r, \frac{\pi}{2}\right) &= 0 & \rightarrow \quad R(r)\Theta\left(\frac{\pi}{2}\right) &= 0 & \rightarrow \quad \Theta\left(\frac{\pi}{2}\right) &= 0 \end{aligned}$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta(0) &= C_1 = 0 \\ \Theta\left(\frac{\pi}{2}\right) &= C_1 \cos\mu\frac{\pi}{2} + C_2 \sin\mu\frac{\pi}{2} = 0 \end{aligned}$$

This first equation makes the second equation reduce to $C_2 \sin\mu\frac{\pi}{2} = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$.

$$\begin{aligned} \sin\mu\frac{\pi}{2} &= 0 \\ \mu\frac{\pi}{2} &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= 2n \end{aligned}$$

There are positive eigenvalues $\lambda = (2n)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_2 \sin\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \sin 2n\theta.$$

Using $\lambda = 4n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 4n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = 4n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - 4n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2m(m-1)r^{m-2} + rmr^{m-1} - 4n^2r^m = 0$$

$$m(m-1)r^m + mr^m - 4n^2r^m = 0$$

$$m(m-1) + m - 4n^2 = 0$$

$$m^2 - 4n^2 = 0$$

$$(m+2n)(m-2n) = 0$$

$$m = \{-2n, 2n\}$$

Two solutions to the ODE are $R = r^{-2n}$ and $R = r^{2n}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-2n} + Br^{2n}$$

Now check to see if zero is an eigenvalue.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta(0) = C_4 = 0$$
$$\Theta\left(\frac{\pi}{2}\right) = C_3\frac{\pi}{2} + C_4 = 0$$

This first equation makes the second equation reduce to $C_3\frac{\pi}{2} = 0$, which makes $C_3 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5\cosh\gamma\theta + C_6\sinh\gamma\theta$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\Theta(0) = C_5 = 0$$
$$\Theta\left(\frac{\pi}{2}\right) = C_5\cosh\gamma\frac{\pi}{2} + C_6\sinh\gamma\frac{\pi}{2} = 0$$

This first equation makes the second equation reduce to $C_6\sinh\gamma\frac{\pi}{2} = 0$. No nonzero value of γ can satisfy this equation, so $C_6 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r,\theta) = \sum_{n=1}^{\infty}(A_n r^{-2n} + B_n r^{2n})\sin 2n\theta$$

For the solution to remain finite as $r \to 0$, set $A_n = 0$.

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n}\sin 2n\theta}$$

Differentiate it with respect to r.

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} 2nB_n r^{2n-1}\sin 2n\theta$$

Use the boundary condition at $r = 1$ to determine the remaining constants B_n.

$$\frac{\partial u}{\partial r}(1,\theta) = \sum_{n=1}^{\infty} 2nB_n \sin 2n\theta = f(\theta)$$

Multiply both sides by $\sin 2p\theta$, where p is an integer.

$$\sum_{n=1}^{\infty} 2nB_n \sin 2n\theta \sin 2p\theta = f(\theta)\sin 2p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2}\sum_{n=1}^{\infty} 2nB_n \sin 2n\theta \sin 2p\theta \, d\theta = \int_0^{\pi/2} f(\theta)\sin 2p\theta \, d\theta$$

Bring the constants in front of the integral.

$$\sum_{n=1}^{\infty} 2nB_n \int_0^{\pi/2} \sin 2n\theta \sin 2p\theta \, d\theta = \int_0^{\pi/2} f(\theta)\sin 2p\theta \, d\theta$$

Because the sine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$2nB_n \int_0^{\pi/2} \sin^2 2n\theta \, d\theta = \int_0^{\pi/2} f(\theta)\sin 2n\theta \, d\theta$$

$$2nB_n\left(\frac{\pi}{4}\right) = \int_0^{\pi/2} f(\theta)\sin 2n\theta \, d\theta$$

Therefore,

$$\boxed{B_n = \frac{2}{n\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta \, d\theta.}$$

Part (d)

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial \theta}(r,0) = 0 \quad &\rightarrow \quad R(r)\Theta'(0) = 0 \quad &\rightarrow \quad \Theta'(0) = 0 \\ \frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad R(r)\Theta'\left(\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad \Theta'\left(\frac{\pi}{2}\right) = 0 \end{aligned}$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin\mu\theta + C_2 \cos\mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta'(0) &= \mu(C_2) = 0 \\ \Theta'\left(\frac{\pi}{2}\right) &= \mu\left(-C_1 \sin\mu\frac{\pi}{2} + C_2 \cos\mu\frac{\pi}{2}\right) = 0 \end{aligned}$$

This first equation gives $C_2 = 0$, which makes the second equation reduce to $-C_1\mu\sin\mu\frac{\pi}{2} = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\begin{aligned} \sin\mu\frac{\pi}{2} &= 0 \\ \mu\frac{\pi}{2} &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= 2n \end{aligned}$$

There are positive eigenvalues $\lambda = (2n)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \cos 2n\theta.$$

Using $\lambda = 4n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 4n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = 4n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - 4n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - 4n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - 4n^2 r^m = 0$$

$$m(m-1) + m - 4n^2 = 0$$

$$m^2 - 4n^2 = 0$$

$$(m+2n)(m-2n) = 0$$

$$m = \{-2n, 2n\}$$

Two solutions to the ODE are $R = r^{-2n}$ and $R = r^{2n}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-2n} + Br^{2n}$$

Now check to see if zero is an eigenvalue.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta'(0) = C_3 = 0$$

$$\Theta'\left(\frac{\pi}{2}\right) = C_3 = 0$$

C_4 remains arbitrary.

$$\Theta(\theta) = C_4$$

The trivial solution is not obtained, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = D_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{D_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = D_1 \ln r + D_2$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh \gamma\theta + C_6 \sinh \gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh \gamma\theta + C_6 \cosh \gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\Theta'(0) = \gamma(C_6) = 0$$
$$\Theta'\left(\frac{\pi}{2}\right) = \gamma\left(C_5 \sinh \gamma\frac{\pi}{2} + C_6 \cosh \gamma\frac{\pi}{2}\right) = 0$$

This first equation gives $C_6 = 0$, which makes the second equation reduce to $C_5\gamma \sinh \gamma\frac{\pi}{2} = 0$. No nonzero value of γ can satisfy this equation, so $C_5 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r,\theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty}(A_n r^{-2n} + B_n r^{2n}) \cos 2n\theta$$

For the solution to remain finite as $r \to 0$, set $A_0 = 0$ and $A_n = 0$.

$$\boxed{u(r,\theta) = B_0 + \sum_{n=1}^{\infty} B_n r^{2n} \cos 2n\theta}$$

Differentiate it with respect to r.

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} 2nB_n r^{2n-1} \cos 2n\theta$$

Use the boundary condition at $r = 1$ to determine B_n.

$$\frac{\partial u}{\partial r}(1,\theta) = \sum_{n=1}^{\infty} 2nB_n \cos 2n\theta = g(\theta)$$

Multiply both sides by $\cos 2p\theta$, where p is an integer.

$$\sum_{n=1}^{\infty} 2nB_n \cos 2n\theta \cos 2p\theta = g(\theta) \cos 2p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2} \sum_{n=1}^{\infty} 2nB_n \cos 2n\theta \cos 2p\theta \, d\theta = \int_0^{\pi/2} g(\theta) \cos 2p\theta \, d\theta$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} 2nB_n \int_0^{\pi/2} \cos 2n\theta \cos 2p\theta \, d\theta = \int_0^{\pi/2} g(\theta) \cos 2p\theta \, d\theta$$

Because the cosine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$2nB_n \int_0^{\pi/2} \cos^2 2n\theta \, d\theta = \int_0^{\pi/2} g(\theta) \cos 2n\theta \, d\theta$$

$$2nB_n \left(\frac{\pi}{4}\right) = \int_0^{\pi/2} g(\theta) \cos 2n\theta \, d\theta$$

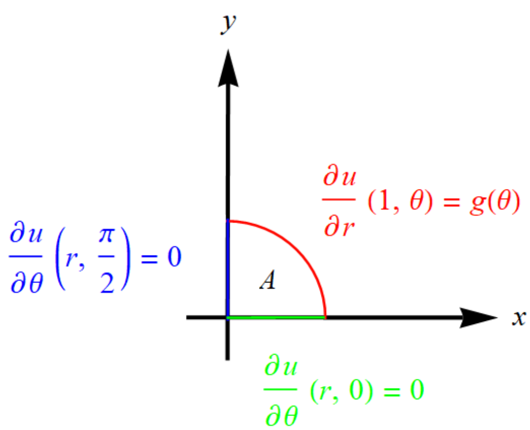
Therefore,

$$\boxed{B_n = \frac{2}{n\pi} \int_0^{\pi/2} g(\theta) \cos 2n\theta \, d\theta.}$$

Understand that $g(\theta)$ is not an arbitrary function. There is a solvability condition that g must satisfy for there to be a solution to the PDE.

$$\nabla^2 u = 0$$

Integrate both sides over the quarter-circle ($0 \leq \theta \leq \pi/2$, $0 \leq r \leq 1$).



This domain in the xy-plane is illustrated above.

$$\iint_A \nabla^2 u \, dA = \iint_A 0 \, dA$$

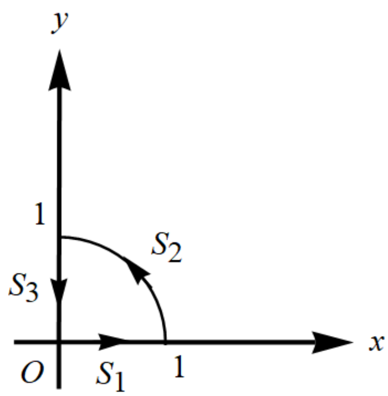
The right side is zero.

$$\iint_A \nabla \cdot \nabla u \, dA = 0$$

Apply the two-dimensional divergence theorem to turn this double integral into a counterclockwise closed loop integral around the quarter-circle's boundary.

$$\oint_S \nabla u \cdot \hat{\mathbf{n}}\, ds = 0$$

Here $\hat{\mathbf{n}}$ is an outward unit vector normal to the boundary, which can be split into three parts, S_1, S_2, and S_3.



The outward normal unit vectors to S_1, S_2, and S_3 are $-\hat{\mathbf{y}}$, $\hat{\mathbf{r}}$, and $-\hat{\mathbf{x}}$, respectively.

$$\int_{S_1} \nabla u \cdot (-\hat{\mathbf{y}})\, ds + \int_{S_2} \nabla u \cdot \hat{\mathbf{r}}\, ds + \int_{S_3} \nabla u \cdot (-\hat{\mathbf{x}})\, ds = 0$$

Evaluate the dot products.

$$\int_{S_1} \left(-\frac{\partial u}{\partial y}\right) ds + \int_{S_2} \frac{\partial u}{\partial r}\, ds + \int_{S_3} \left(-\frac{\partial u}{\partial x}\right) ds = 0$$

Bring the minus signs in front.

$$-\int_{S_1} \frac{\partial u}{\partial y}\, ds + \int_{S_2} \frac{\partial u}{\partial r}\, ds - \int_{S_3} \frac{\partial u}{\partial x}\, ds = 0$$

Along S_1, u_y is evaluated at $y = 0$; along S_2, u_r is evaluated at $r = 1$; and along S_3, u_x is evaluated at $x = 0$.

$$-\int_{S_1} \frac{\partial u}{\partial y}\bigg|_{y=0} ds + \int_{S_2} \frac{\partial u}{\partial r}\bigg|_{r=1} ds - \int_{S_3} \frac{\partial u}{\partial x}\bigg|_{x=0} ds = 0$$

The differential of arc length ds is always positive regardless of whether the path around the boundary is clockwise or counterclockwise. So don't mind the orientation when parameterizing the integration paths.

$$-\int_0^1 \frac{\partial u}{\partial y}\bigg|_{y=0} dr + \int_0^{\pi/2} \frac{\partial u}{\partial r}\bigg|_{r=1} d\theta - \int_0^1 \frac{\partial u}{\partial x}\bigg|_{x=0} dr = 0 \tag{2}$$

We know that $x = r\cos\theta$ and $y = r\sin\theta$. Solving this system of equations for r and θ yields

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\frac{y}{x}.$$

Use the chain rule to write $\partial u/\partial x$ in terms of r and θ.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial u}{\partial r}\left[\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x\right] + \frac{\partial u}{\partial \theta}\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)\right] \\ &= \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial u}{\partial \theta}\left(\frac{y}{x^2+y^2}\right) \\ &= \frac{\partial u}{\partial r}(\cos\theta) - \frac{\partial u}{\partial \theta}\left(\frac{\sin\theta}{r}\right)\end{aligned}$$

$x = 0$ corresponds to $\theta = \pi/2$, so

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\left[\frac{\partial u}{\partial r}(\cos\theta) - \frac{\partial u}{\partial \theta}\left(\frac{\sin\theta}{r}\right)\right]\right|_{\theta=\pi/2} = -\frac{1}{r}\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2}.$$

Use the chain rule to write $\partial u/\partial y$ in terms of r and θ.

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{\partial u}{\partial r}\left[\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y\right] + \frac{\partial u}{\partial \theta}\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\right] \\ &= \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right) \\ &= \frac{\partial u}{\partial r}(\sin\theta) + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\end{aligned}$$

$y = 0$ corresponds to $\theta = 0$, so

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \left.\left[\frac{\partial u}{\partial r}(\sin\theta) + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right]\right|_{\theta=0} = \frac{1}{r}\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0}.$$

Consequently, equation (2) becomes

$$-\int_0^1 \frac{1}{r}\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} dr + \int_0^{\pi/2} \left.\frac{\partial u}{\partial r}\right|_{r=1} d\theta - \int_0^1 \left.\left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\right|_{\theta=\pi/2} dr = 0.$$

Substitute the prescribed boundary conditions, $u_\theta(r,0) = 0$ and $u_\theta(r,\pi/2) = 0$ and $u_r(1,\theta) = g(\theta)$.

$$-\int_0^1 0\,dr + \int_0^{\pi/2} g(\theta)\,d\theta - \int_0^1 0\,dr = 0$$

Therefore,

$$\int_0^{\pi/2} g(\theta)\,d\theta = 0.$$

This is the solvability condition. If u represents the steady-state temperature, then the spatial derivatives of u (the boundary conditions) physically signify heat fluxes. Since no heat leaves the quarter-circle along $\theta = 0$ and $\theta = \pi/2$, the net heat flux entering along $r = 1$ better be zero or else the temperature inside will rise or drop indefinitely. There will be no steady state if the solvability condition is not satisfied.