# Exercise 3.2.3

Show that the Fourier series operation is linear; that is, show that the Fourier series of $c_1 f(x) + c_2 g(x)$ is the sum of $c_1$ times the Fourier series of $f(x)$ and $c_2$ times the Fourier series of $g(x)$.

---

### Solution

Suppose that $f(x)$ and $g(x)$ are defined on $-L \leq x \leq L$ and have their own Fourier series expansions.

$$f(x) = A_0' + \sum_{n=1}^{\infty} \left( A_n' \cos \frac{n\pi x}{L} + B_n' \sin \frac{n\pi x}{L} \right) \qquad g(x) = A_0'' + \sum_{n=1}^{\infty} \left( A_n'' \cos \frac{n\pi x}{L} + B_n'' \sin \frac{n\pi x}{L} \right),$$

where

$$\begin{aligned} A_0' &= \frac{1}{2L} \int_{-L}^{L} f(x)\,dx & A_0'' &= \frac{1}{2L} \int_{-L}^{L} g(x)\,dx \\ A_n' &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx & A_n'' &= \frac{1}{L} \int_{-L}^{L} g(x) \cos \frac{n\pi x}{L}\,dx \\ B_n' &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx & B_n'' &= \frac{1}{L} \int_{-L}^{L} g(x) \sin \frac{n\pi x}{L}\,dx. \end{aligned}$$

Now write the Fourier series expansion of $c_1 f(x) + c_2 g(x)$.

$$c_1 f(x) + c_2 g(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \tag{1}$$

To determine $A_0$, integrate both sides with respect to $x$ from $-L$ to $L$.

$$\int_{-L}^{L} [c_1 f(x) + c_2 g(x)]\,dx = \int_{-L}^{L} \left[ A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \right] dx$$

Split up the integrals on both sides and bring the constants in front.

$$c_1 \int_{-L}^{L} f(x)\,dx + c_2 \int_{-L}^{L} g(x)\,dx = A_0 \underbrace{\int_{-L}^{L} dx}_{=\,2L} + \sum_{n=1}^{\infty} \left( A_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L}\,dx}_{=\,0} + B_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L}\,dx}_{=\,0} \right)$$

Evaluate the integrals.

$$c_1 \int_{-L}^{L} f(x)\,dx + c_2 \int_{-L}^{L} g(x)\,dx = A_0(2L)$$

Consequently, $A_0$ is a linear combination of $A_0'$ and $A_0''$.

$$\begin{aligned} A_0 &= \frac{c_1}{2L} \int_{-L}^{L} f(x)\,dx + \frac{c_2}{2L} \int_{-L}^{L} g(x)\,dx \\ &= c_1 A_0' + c_2 A_0'' \end{aligned}$$

To determine $A_n$, multiply both sides of equation (1) by $\cos\frac{p\pi x}{L}$

$$c_1 f(x)\cos\frac{p\pi x}{L} + c_2 g(x)\cos\frac{p\pi x}{L} = A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + B_n\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)$$

and then integrate both sides with respect to $x$ from $-L$ to $L$.

$$\begin{aligned}\int_{-L}^{L}&\left[c_1 f(x)\cos\frac{p\pi x}{L} + c_2 g(x)\cos\frac{p\pi x}{L}\right]dx\\ &= \int_{-L}^{L}\left[A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + B_n\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)\right]dx\end{aligned}$$

Split up the integrals on both sides and bring the constants in front.

$$\begin{aligned}c_1&\int_{-L}^{L} f(x)\cos\frac{p\pi x}{L}\,dx + c_2\int_{-L}^{L} g(x)\cos\frac{p\pi x}{L}\,dx\\ &= A_0\underbrace{\int_{-L}^{L}\cos\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty}\left(A_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx + B_n\underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx}_{=\,0}\right)\end{aligned}$$

Because the sine and cosine functions are orthogonal, the third integral on the right side is zero. Also, the second integral on the right side is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$c_1\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + c_2\int_{-L}^{L} g(x)\cos\frac{n\pi x}{L}\,dx = A_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx$$

Evaluate the integral.

$$c_1\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + c_2\int_{-L}^{L} g(x)\cos\frac{n\pi x}{L}\,dx = A_n(L)$$

Consequently, $A_n$ is a linear combination of $A_n'$ and $A_n''$.

$$\begin{aligned}A_n &= \frac{c_1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{c_2}{L}\int_{-L}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\\ &= c_1 A_n' + c_2 A_n''\end{aligned}$$

To determine $B_n$, multiply both sides of equation (1) by $\sin\frac{p\pi x}{L}$

$$c_1 f(x)\sin\frac{p\pi x}{L} + c_2 g(x)\sin\frac{p\pi x}{L} = A_0\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + B_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)$$

and then integrate both sides with respect to $x$ from $-L$ to $L$.

$$\begin{aligned}\int_{-L}^{L}&\left[c_1 f(x)\sin\frac{p\pi x}{L} + c_2 g(x)\sin\frac{p\pi x}{L}\right]dx \\ &= \int_{-L}^{L}\left[A_0\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + B_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)\right]dx\end{aligned}$$

Split up the integrals on both sides and bring the constants in front.

$$\begin{aligned}c_1&\int_{-L}^{L} f(x)\sin\frac{p\pi x}{L}\,dx + c_2\int_{-L}^{L} g(x)\sin\frac{p\pi x}{L}\,dx \\ &= A_0\underbrace{\int_{-L}^{L}\sin\frac{p\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty}\left(A_n\underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx}_{=0} + B_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx\right)\end{aligned}$$

Because the sine and cosine functions are orthogonal, the second integral on the right side is zero. Also, the third integral on the right side is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$c_1\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + c_2\int_{-L}^{L} g(x)\sin\frac{n\pi x}{L}\,dx = B_n\int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx$$

Evaluate the integral.

$$c_1\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + c_2\int_{-L}^{L} g(x)\sin\frac{n\pi x}{L}\,dx = B_n(L)$$

Consequently, $B_n$ is a linear combination of $B_n'$ and $B_n''$.

$$\begin{aligned}B_n &= \frac{c_1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{c_2}{L}\int_{-L}^{L} g(x)\sin\frac{n\pi x}{L}\,dx \\ &= c_1 B_n' + c_2 B_n''\end{aligned}$$

Therefore, because the integral is a linear operator, the Fourier series operation is linear.