Exercise 3.3.8

(a) Determine formulas for the even extension of any $f(x)$. Compare to the formula for the even part of $f(x)$.

(b) Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

(c) Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. What occurs? Similarly, add the even and odd extensions. What occurs then?

Solution

Part (a)

For any function $f(x)$, the even part is

$$\text{Even Part:} \quad \frac{f(x) + f(-x)}{2}.$$

For a function $f(x)$ defined on $0 < x < \infty$, the even extension to the whole line $(-\infty < x < \infty)$ is

$$\text{Even Extension:} \quad \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.$$

Part (b)

For any function $f(x)$, the odd part is

$$\text{Odd Part:} \quad \frac{f(x) - f(-x)}{2}.$$

For a function $f(x)$ defined on $0 < x < \infty$, the odd extension to the whole line is

$$\text{Odd Extension:} \quad \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}.$$

Part (c)

For this prescribed function,

$$\text{Even Part:} \quad \begin{cases} \frac{1}{2}[x + (-x)^2] & x > 0 \\ \frac{1}{2}[x^2 + (-x)] & x < 0 \end{cases} \qquad \text{Even Extension:} \quad \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$\text{Odd Part:} \quad \begin{cases} \frac{1}{2}[x - (-x)^2] & x > 0 \\ \frac{1}{2}[x^2 - (-x)] & x < 0 \end{cases} \qquad \text{Odd Extension:} \quad \begin{cases} x & x > 0 \\ -(-x) & x < 0 \end{cases}.$$

Simplifying these expressions gives

$$\text{Even Part:} \quad \begin{cases} \frac{1}{2}(x+x^2) & x>0 \\ \frac{1}{2}(x^2-x) & x<0 \end{cases} \qquad \text{Even Extension:} \quad \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$$

$$\text{Odd Part:} \quad \begin{cases} \frac{1}{2}(x-x^2) & x>0 \\ \frac{1}{2}(x^2+x) & x<0 \end{cases} \qquad \text{Odd Extension:} \quad \begin{cases} x & x>0 \\ x & x<0 \end{cases}.$$

Adding the even and odd parts results in the original function,

$$\begin{cases} x & x>0 \\ x^2 & x<0 \end{cases},$$

while adding the even and odd extensions results in

$$\begin{cases} 2x & x>0 \\ 0 & x<0 \end{cases}.$$