Exercise 3.4.2

Suppose that $f(x)$ and df/dx are piecewise smooth. Prove that the Fourier series of $f(x)$ can be differentiated term by term if the Fourier series of $f(x)$ is continuous.

Solution

The fact that $f(x)$ and df/dx are piecewise smooth (on the interval $-L \leq x \leq L$) means that they each have a Fourier series representation.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \tag{1}$$

$$\frac{df}{dx} = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi x}{L} + D_n \sin \frac{n\pi x}{L} \right) \tag{2}$$

The coefficients in equation (1) are known.

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

The aim is to show that

$$C_0 = 0 \quad \text{and} \quad C_n = \frac{n\pi}{L} B_n \quad \text{and} \quad D_n = -\frac{n\pi}{L} A_n.$$

To get C_0, integrate both sides of equation (2) with respect to x from $-L$ to L.

$$\int_{-L}^{L} \frac{df}{dx}\, dx = \int_{-L}^{L} \left[C_0 + \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi x}{L} + D_n \sin \frac{n\pi x}{L} \right) \right] dx$$

Evaluate the integral on the left, split up the integral on the right, and bring the constants in front.

$$f(L) - f(-L) = C_0 \underbrace{\int_{-L}^{L} dx}_{=\,2L} + \sum_{n=1}^{\infty} \left(C_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx}_{=\,0} + D_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L}\, dx}_{=\,0} \right)$$

If the Fourier series of $f(x)$ is continuous, then $f(L) = f(-L)$.

$$0 = C_0(2L)$$

Therefore,

$$\boxed{C_0 = 0.}$$

To get C_n, multiply both sides of equation (2) by $\cos\frac{p\pi x}{L}$, where p is an integer,

$$\frac{df}{dx}\cos\frac{p\pi x}{L} = C_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + D_n\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L}\frac{df}{dx}\cos\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left[C_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + D_n\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)\right]dx \\ &= C_0\underbrace{\int_{-L}^{L}\cos\frac{p\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty}\left(C_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx + D_n\underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx}_{=0}\right)\end{aligned}$$

The third integral on the right is zero because the sine and cosine functions are orthogonal. The second integral only yields a nonzero result if $n = p$.

$$\int_{-L}^{L}\frac{df}{dx}\cos\frac{n\pi x}{L}\,dx = C_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx$$

Use integration by parts on the left and evaluate the integral on the right.

$$f(x)\cos\frac{n\pi x}{L}\bigg|_{-L}^{L} - \int_{-L}^{L}f(x)\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)dx = C_n(L)$$

$$f(L)\cos n\pi - f(-L)\cos(-n\pi) - \int_{-L}^{L}f(x)\left(-\frac{n\pi}{L}\sin\frac{n\pi x}{L}\right)dx = C_n(L)$$

$$[f(L) - f(-L)]\cos n\pi + n\pi\left[\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx\right] = C_n(L)$$

If the Fourier series of $f(x)$ is continuous, then $f(L) = f(-L)$.

$$n\pi(B_n) = C_n(L)$$

Therefore,

$$\boxed{C_n = \frac{n\pi}{L}B_n.}$$

To get D_n, multiply both sides of equation (2) by $\sin\frac{p\pi x}{L}$, where p is an integer,

$$\frac{df}{dx}\sin\frac{p\pi x}{L} = C_0\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + D_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L}\frac{df}{dx}\sin\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left[C_0\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + D_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)\right]dx \\ &= C_0\underbrace{\int_{-L}^{L}\sin\frac{p\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty}\left(C_n\underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx}_{=0} + D_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx\right)\end{aligned}$$

The second integral on the right is zero because the sine and cosine functions are orthogonal. The third integral only yields a nonzero result if $n = p$.

$$\int_{-L}^{L} \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx = D_n \int_{-L}^{L} \sin^2 \frac{n\pi x}{L}\, dx$$

Use integration by parts on the left and evaluate the integral on the right.

$$f(x) \sin \frac{n\pi x}{L}\bigg|_{-L}^{L} - \int_{-L}^{L} f(x) \frac{d}{dx}\left(\sin \frac{n\pi x}{L}\right) dx = D_n(L)$$

$$f(L) \sin n\pi - f(-L) \sin(-n\pi) - \int_{-L}^{L} f(x) \left(\frac{n\pi}{L} \cos \frac{n\pi x}{L}\right) dx = D_n(L)$$

$$[f(L) + f(-L)] \sin n\pi - n\pi \left[\frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx\right] = D_n(L)$$

$$-n\pi(A_n) = D_n(L)$$

Therefore,

$$\boxed{D_n = -\frac{n\pi}{L} A_n.}$$

The Fourier series of $f(x)$ can be differentiated term by term if the Fourier series of $f(x)$ is continuous. But even if it's not continuous, the Fourier series of df/dx can still be written using the following formulas.

$$C_0 = \frac{1}{2L}[f(L) - f(-L)]$$

$$C_n = \frac{(-1)^n}{L}[f(L) - f(-L)] + \frac{n\pi}{L} B_n$$

$$D_n = -\frac{n\pi}{L} A_n$$