Exercise 3.4.6

There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}. \tag{3.4.22}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\frac{n\pi x}{L},$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos\frac{n\pi x}{L}. \tag{3.4.23}$$

Since Equations (3.4.22) and (3.4.23) give the Fourier cosine series of e^x, they must be identical. Thus,

$$\left.\begin{aligned} A_0 &= 0 \\ A_n &= 0 \end{aligned}\right\} \quad \text{(obviously wrong!)}.$$

By correcting the mistakes, you should be able to obtain A_0 and A_n *without* using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L\, dx$.

Solution

e^x is a continuous function on $0 \leq x \leq L$, so it has a Fourier cosine series expansion.

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \tag{1}$$

Because e^x is continuous, there's no problem differentiating its cosine series with respect to x term by term.

$$e^x = \sum_{n=1}^{\infty} \left(-\frac{n\pi}{L} A_n\right) \sin\frac{n\pi x}{L}$$

This is now a sine series, so differentiating term by term is not justified because $e^0 \neq 0$ and $e^L \neq 0$. Rather, use Eq. 3.4.13 on page 117.

$$e^x = \frac{1}{L}(e^L - 1) + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L}\left(-\frac{n\pi}{L} A_n\right) + \frac{2}{L}[(-1)^n e^L - 1]\right] \cos\frac{n\pi x}{L} \tag{2}$$

Comparing equations (1) and (2) gives

$$A_0 = \frac{1}{L}(e^L - 1)$$

$$A_n = \frac{n\pi}{L}\left(-\frac{n\pi}{L} A_n\right) + \frac{2}{L}[(-1)^n e^L - 1] \quad \rightarrow \quad A_n = \frac{2L[(-1)^n e^L - 1]}{n^2\pi^2 + L^2}.$$