Problem 1.13

Expressing one vector in terms of another
Let $\mathbf{A}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$.

Solution

Suppose that $\mathbf{A} = \langle A_1, A_2, A_3\rangle$ and $\mathbf{n} = \langle n_1, n_2, n_3\rangle$. Let the unit vector in the x-direction be denoted as $\boldsymbol{\delta}_1$, let the unit vector in the y-direction be denoted as $\boldsymbol{\delta}_2$, and let the unit vector in the z-direction be denoted as $\boldsymbol{\delta}_3$.

$$\begin{aligned}
(\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}} &= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \left[\left(\sum_{i=1}^{3} n_i\boldsymbol{\delta}_i\right)\times\left(\sum_{j=1}^{3} A_j\boldsymbol{\delta}_j\right)\right]\times\left(\sum_{k=1}^{3} n_k\boldsymbol{\delta}_k\right) \\
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \left[\sum_{i=1}^{3}\sum_{j=1}^{3} n_i A_j(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)\right]\times\left(\sum_{k=1}^{3} n_k\boldsymbol{\delta}_k\right) \\
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \left[\sum_{i=1}^{3}\sum_{j=1}^{3} n_i A_j\left(\sum_{l=1}^{3}\varepsilon_{ijl}\boldsymbol{\delta}_l\right)\right]\times\left(\sum_{k=1}^{3} n_k\boldsymbol{\delta}_k\right)
\end{aligned}$$

The cross product has been written in terms of the Levi-Civita symbol ε, which is defined as

$$\varepsilon_{ijl} = \begin{cases} 1 & \text{if } (i,j,l) \text{ is } (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,l) \text{ is } (3,2,1) \text{ or } (1,3,2) \text{ or } (2,1,3) \\ 0 & \text{if } i=j \text{ or } j=l \text{ or } i=l \end{cases}.$$

$$\begin{aligned}
(\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}} &= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3} n_i A_j\varepsilon_{ijl}\boldsymbol{\delta}_l\right)\times\left(\sum_{k=1}^{3} n_k\boldsymbol{\delta}_k\right) \\
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} n_i n_k A_j\varepsilon_{ijl}(\boldsymbol{\delta}_l\times\boldsymbol{\delta}_k) \\
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} n_i n_k A_j\varepsilon_{ijl}\left(\sum_{m=1}^{3}\varepsilon_{lkm}\boldsymbol{\delta}_m\right) \\
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} n_i n_k A_j\varepsilon_{ijl}\varepsilon_{lkm}\boldsymbol{\delta}_m
\end{aligned}$$

Cyclically permute the indices of the second Levi-Civita symbol so that l is in the third position.

$$\begin{aligned}
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} n_i n_k A_j\varepsilon_{ijl}\varepsilon_{kml}\boldsymbol{\delta}_m \\
&= (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} n_i n_k A_j\left(\sum_{l=1}^{3}\varepsilon_{ijl}\varepsilon_{kml}\right)\boldsymbol{\delta}_m
\end{aligned}$$

Since we have a product of two Levi-Civita symbols, the sum over l can be expressed in terms of the Kronecker delta function, defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

by using a known property.

$$\begin{aligned}
(\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}} &= (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} n_i n_k A_j (\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk})\boldsymbol{\delta}_m \\
&= (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} n_i n_k A_j (\delta_{ik}\delta_{jm})\boldsymbol{\delta}_m \\
&\qquad - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} n_i n_k A_j (\delta_{im}\delta_{jk})\boldsymbol{\delta}_m
\end{aligned}$$

δ_{jm} makes $j = m$ in the first quadruple sum, and δ_{jk} makes $j = k$ in the second quadruple sum.

$$= (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} n_i n_k A_j (\delta_{ik})\boldsymbol{\delta}_j - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3} n_i n_j A_j (\delta_{im})\boldsymbol{\delta}_m$$

δ_{ik} makes $i = k$ in the first triple sum, and δ_{im} makes $i = m$ in the second triple sum.

$$\begin{aligned}
&= (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \sum_{i=1}^{3}\sum_{j=1}^{3} n_i^2 A_j \boldsymbol{\delta}_j - \sum_{i=1}^{3}\sum_{j=1}^{3} n_i n_j A_j \boldsymbol{\delta}_i \\
&= (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \left(\sum_{i=1}^{3} n_i^2\right)\left(\sum_{j=1}^{3} A_j \boldsymbol{\delta}_j\right) - \left(\sum_{j=1}^{3} A_j n_j\right)\left(\sum_{i=1}^{3} n_i \boldsymbol{\delta}_i\right)
\end{aligned}$$

The sum over i of n_i^2 is 1 because $\hat{\mathbf{n}}$ is a unit vector—it has unit magnitude.

$$\begin{aligned}
&= (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (1)\mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\
&= \mathbf{A}
\end{aligned}$$

Therefore,

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}}.$$