# Problem 1.23

*Smooth elevator ride*

For a smooth ("low jerk") ride, an elevator is programmed to start from rest and accelerate according to

$$a(t) = (a_m/2)[1 - \cos(2\pi t/T)] \qquad 0 \le t \le T$$
$$a(t) = -(a_m/2)[1 - \cos(2\pi t/T)] \qquad T \le t \le 2T$$

where $a_m$ is the maximum acceleration and $2T$ is the total time for the trip.

(a) Draw sketches of $a(t)$ and the jerk as functions of time.

(b) What is the elevator's maximum speed?

(c) Find an approximate expression for the speed at short times near the start of the ride, $t \ll T$.

(d) What is the time required for a trip of distance $D$?

[TYPO: The right parenthesis is missing.]

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### Solution

Jerk $j(t)$ is the rate that the acceleration changes with respect to time.

$$j(t) = \frac{da}{dt}$$

Since

$$a(t) = \begin{cases} \frac{a_m}{2}\left(1 - \cos\frac{2\pi t}{T}\right) & 0 \le t \le T \\ -\frac{a_m}{2}\left(1 - \cos\frac{2\pi t}{T}\right) & T \le t \le 2T \end{cases},$$

we have

$$j(t) = \begin{cases} \frac{a_m \pi}{T}\sin\frac{2\pi t}{T} & 0 \le t \le T \\ -\frac{a_m \pi}{T}\sin\frac{2\pi t}{T} & T \le t \le 2T \end{cases}.$$

The velocity $v(t)$ is obtained by integrating the acceleration $a(t)$.

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \begin{cases} \frac{a_m}{2}\left(t - \frac{T}{2\pi}\sin\frac{2\pi t}{T}\right) + C_1 & 0 \le t \le T \\ -\frac{a_m}{2}\left(t - \frac{T}{2\pi}\sin\frac{2\pi t}{T}\right) + C_2 & T \le t \le 2T \end{cases} \end{aligned}$$

The constants of integration, $C_1$ and $C_2$, are determined from the facts that the elevator starts from rest and that the velocity must be continuous at $t = T$.

$$\begin{aligned} v(0) = 0 \quad &\rightarrow \quad 0 + C_1 = 0 \quad &&\rightarrow \quad C_1 = 0 \\ v(T-) = v(T+) \quad &\rightarrow \quad \frac{a_m}{2}(T) = -\frac{a_m}{2}(T) + C_2 \quad &&\rightarrow \quad C_2 = a_m T \end{aligned}$$

Therefore,

$$v(t) = \begin{cases} \frac{a_m}{2}\left(t - \frac{T}{2\pi}\sin\frac{2\pi t}{T}\right) & 0 \le t \le T \\ -\frac{a_m}{2}\left(t - \frac{T}{2\pi}\sin\frac{2\pi t}{T}\right) + a_m T & T \le t \le 2T \end{cases}.$$

The position is obtained by integrating the velocity.

$$\begin{aligned} y(t) &= \int v(t)\,dt \\ &= \begin{cases} \frac{a_m}{2}\left(\frac{t^2}{2} + \frac{T^2}{4\pi^2}\cos\frac{2\pi t}{T}\right) + C_3 & 0 \le t \le T \\ -\frac{a_m}{2}\left(\frac{t^2}{2} + \frac{T^2}{4\pi^2}\cos\frac{2\pi t}{T}\right) + C_4 & T \le t \le 2T \end{cases} \end{aligned}$$

The constants of integration, $C_3$ and $C_4$, are determined from the facts that the elevator starts at $y = 0$ and that the position must be continuous at $t = T$.

$$y(0) = 0 \quad \to \quad \frac{a_m}{2}\left(0 + \frac{T^2}{4\pi^2}\right) + C_3 = 0 \quad \to \quad C_3 = -\frac{a_m T^2}{8\pi^2}$$

$$y(T-) = y(T+) \quad \to \quad \frac{a_m}{2}\left(\frac{T^2}{2} + \frac{T^2}{4\pi^2}\right) - \frac{a_m T^2}{8\pi^2} = -\frac{a_m}{2}\left(\frac{T^2}{2} + \frac{T^2}{4\pi^2}\right) + C_4 \quad \to \quad C_4 = \frac{a_m T^2(1 - 4\pi^2)}{8\pi^2}$$

Therefore,

$$y(t) = \begin{cases} \frac{a_m}{2}\left(\frac{t^2}{2} + \frac{T^2}{4\pi^2}\cos\frac{2\pi t}{T}\right) - \frac{a_m T^2}{8\pi^2} & 0 \le t \le T \\ -\frac{a_m}{2}\left(\frac{t^2}{2} + \frac{T^2}{4\pi^2}\cos\frac{2\pi t}{T}\right) + \frac{a_m T^2(1 - 4\pi^2)}{8\pi^2} & T \le t \le 2T \end{cases}.$$

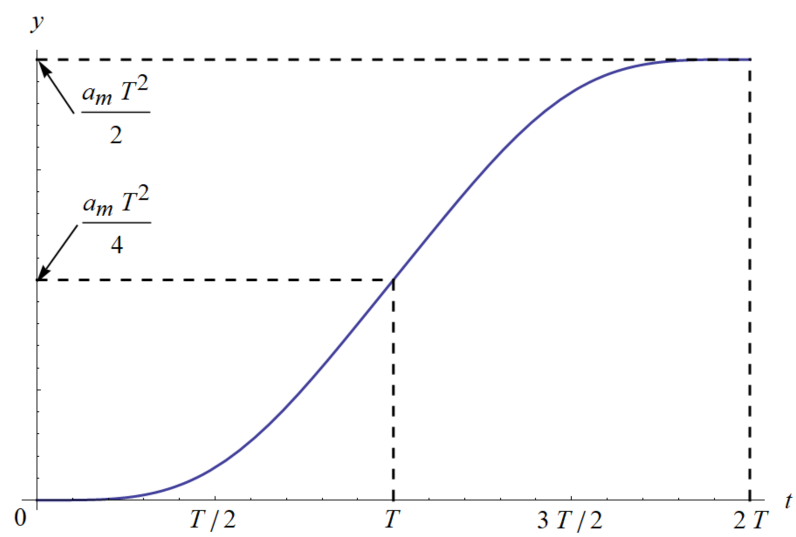


Figure 1: This is a plot of the position $y$ as a function of $t$ for $0 \le t \le 2T$.

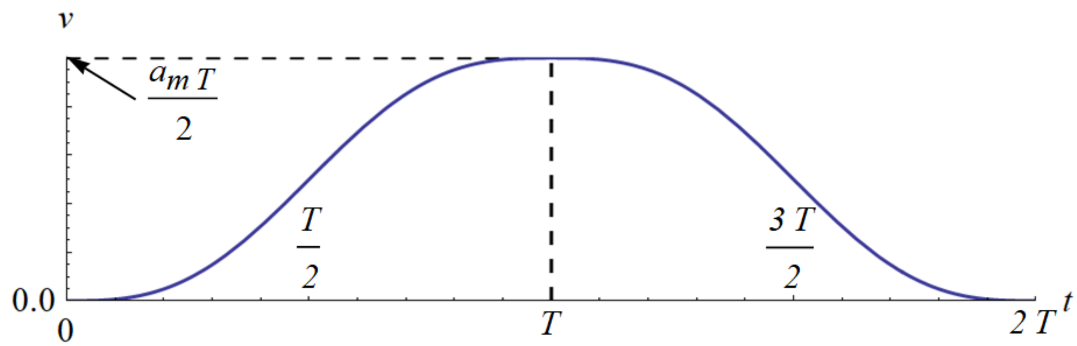


Figure 2: This is a plot of the velocity $v$ as a function of $t$ for $0 \le t \le 2T$.

**Part (a)**

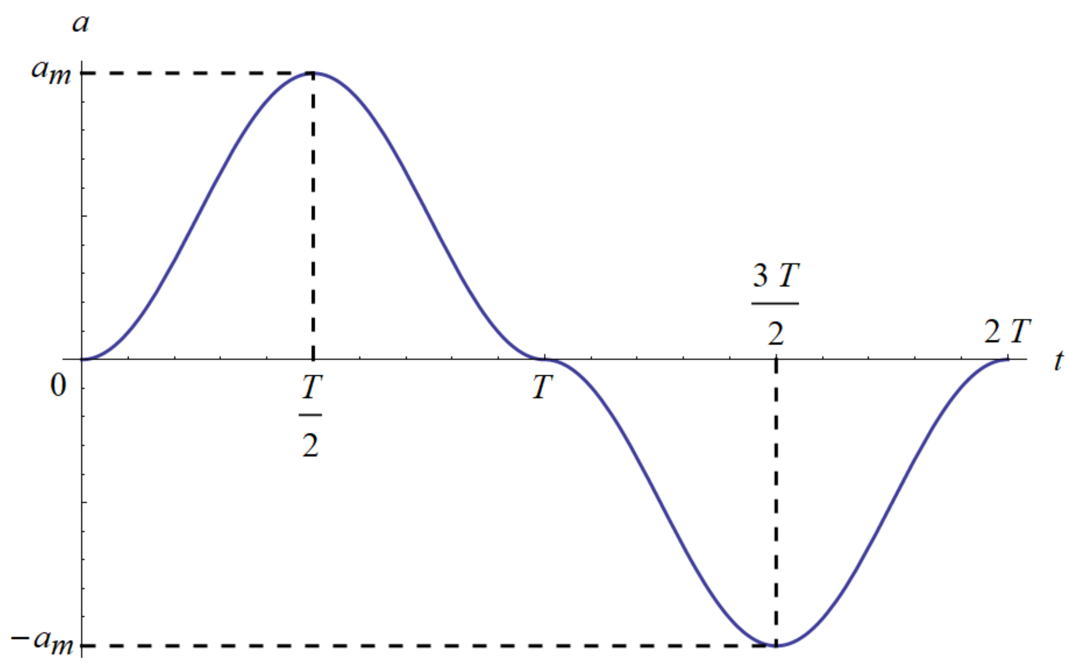


Figure 3: This is a plot of the acceleration $a$ as a function of $t$ for $0 \le t \le 2T$.

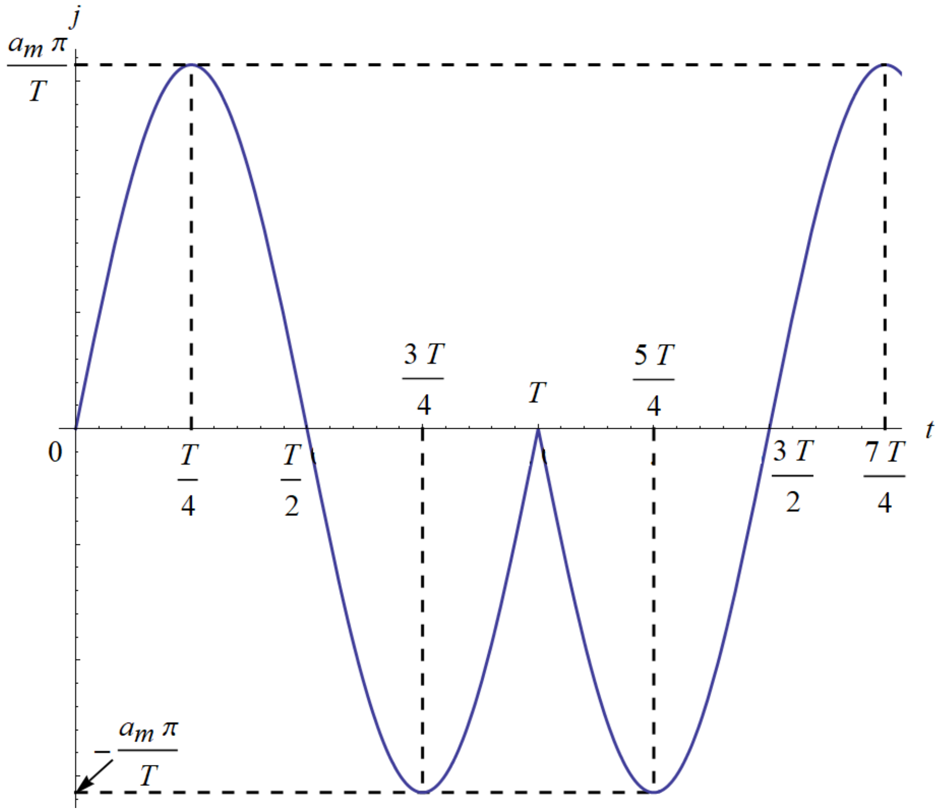


Figure 4: This is a plot of the jerk $j$ as a function of $t$ for $0 \leq t \leq 2T$.

## Part (b)

As indicated in Figure 2, the maximum velocity is

$$v_{\max} = \frac{a_m T}{2}.$$

## Part (c)

Consider the formula for the velocity that's valid for $0 \leq t \leq T$.

$$v(t) = \frac{a_m}{2}\left(t - \frac{T}{2\pi}\sin\frac{2\pi t}{T}\right)$$

The Taylor series expansion for $\sin t$ centered at $t = 0$ is

$$\sin t = t - \frac{t^3}{6} + \cdots,$$

so

$$\sin\frac{2\pi t}{T} = \frac{2\pi t}{T} - \frac{1}{6}\frac{8\pi^3 t^3}{T^3} + \cdots$$

Substitute this into the formula.

$$\begin{aligned} v(t) &= \frac{a_m}{2}\left[t - \frac{T}{2\pi}\left(\frac{2\pi t}{T} - \frac{1}{6}\frac{8\pi^3 t^3}{T^3} + \cdots\right)\right] \\ &= \frac{a_m}{2}\left(\not{t} - \not{t} + \frac{1}{6}\frac{4\pi^2 t^3}{T^2} - \cdots\right) \\ &= \frac{a_m \pi^2 t^3}{3T^2} - \cdots \end{aligned}$$

Therefore, an approximate expression for the speed of the elevator near $t = 0$ is

$$v(t) \approx \frac{a_m \pi^2 t^3}{3T^2}.$$

**Part (d)**

As indicated in Figure 1, the maximum distance traveled by the elevator is $a_m T^2/2$. Set this equal to $D$.

$$\frac{a_m T^2}{2} = D$$

Solve this for $T$.

$$T^2 = \frac{2D}{a_m}$$

$$T = \sqrt{\frac{2D}{a_m}}$$

The total time for an elevator trip is $2T$. Therefore,

$$\text{Total time} = 2\sqrt{\frac{2D}{a_m}} = \sqrt{\frac{8D}{a_m}}.$$