Problem 1.6

Diagonals of a parallelogram
Show that the diagonals of an equilateral parallelogram are perpendicular.

Solution

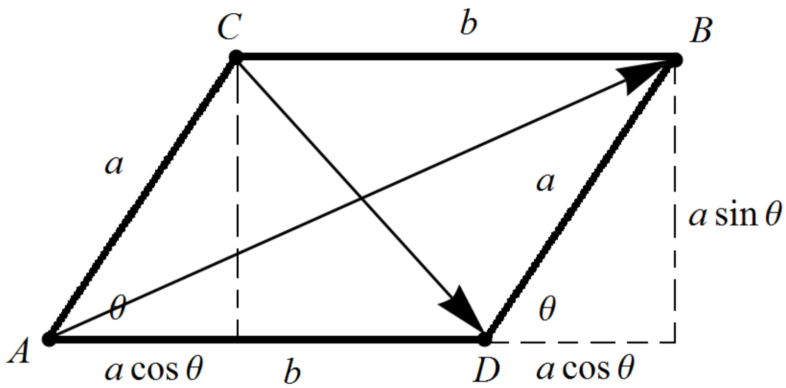

Figure 1: Schematic of a parallelogram.

The diagonals of the parallelogram are represented by vectors $\mathbf{r}_1 = \overrightarrow{AB}$ and $\mathbf{r}_2 = \overrightarrow{CD}$. Our aim here is to show that $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$. From this we'll be able to conclude that the vectors are perpendicular. The positions of points A, B, C, and D are

$$\begin{aligned} A &= (0,0) \\ B &= (b + a\cos\theta, a\sin\theta) \\ C &= (a\cos\theta, a\sin\theta) \\ D &= (b, 0). \end{aligned}$$

The vectors of the diagonals are obtained by taking the position vector of the point where the arrow is headed and subtracting from it the position vector of the point where the arrow came from.

$$\begin{aligned} \mathbf{r}_1 &= \langle b + a\cos\theta, a\sin\theta\rangle - \langle 0,0\rangle = \langle b + a\cos\theta, a\sin\theta\rangle \\ \mathbf{r}_2 &= \langle b, 0\rangle - \langle a\cos\theta, a\sin\theta\rangle = \langle b - a\cos\theta, -a\sin\theta\rangle \end{aligned}$$

Now take the dot product of these two vectors.

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{r}_2 &= (b + a\cos\theta)(b - a\cos\theta) + (a\sin\theta)(-a\sin\theta) \\ \mathbf{r}_1 \cdot \mathbf{r}_2 &= b^2 - a^2\cos^2\theta - a^2\sin^2\theta \\ \mathbf{r}_1 \cdot \mathbf{r}_2 &= b^2 - a^2(\cos^2\theta + \sin^2\theta) \\ \mathbf{r}_1 \cdot \mathbf{r}_2 &= b^2 - a^2 \end{aligned}$$

The fact that the parallelogram is equilateral means that $a = b$. Hence,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0.$$

The left side can be written in terms of the angle between the vectors.

$$|\mathbf{r}_1||\mathbf{r}_2| \cos\theta = 0$$

The magnitudes are nonzero, so the cosine of the angle between the vectors must be 0.

$$\cos\theta = 0$$

Taking the inverse cosine of both sides gives the angle between the vectors.

$$\theta = \frac{\pi}{2}$$

Therefore, the diagonals of an equilateral parallelogram are perpendicular.