Problem 1.8

Vector proof of a trigonometric identity
Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the x–y plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$, and using vector algebra prove that

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Solution

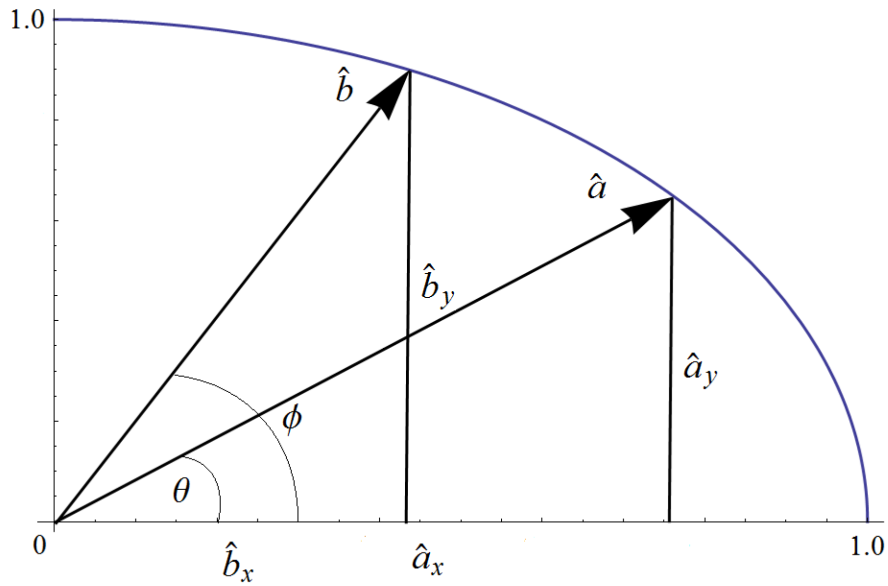

Figure 1: This figure shows the two unit vectors, $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, and their components in the x and y directions.

The cosine is ratio of the adjacent side to the hypotenuse, so we have

$$\cos\theta = \frac{a_x}{|\hat{\mathbf{a}}|} = \frac{a_x}{1} = a_x$$

and

$$\cos\phi = \frac{b_x}{|\hat{\mathbf{b}}|} = \frac{b_x}{1} = b_x.$$

On the other hand, sine is the ratio of the opposite side to the hypotenuse, so we have

$$\sin\theta = \frac{a_y}{|\hat{\mathbf{a}}|} = \frac{a_y}{1} = a_y$$

and

$$\sin\phi = \frac{b_y}{|\hat{\mathbf{b}}|} = \frac{b_y}{1} = b_y.$$

The unit vectors are defined as

$$\hat{\mathbf{a}} = \langle a_x, a_y \rangle$$

and

$$\hat{\mathbf{b}} = \langle b_x, b_y \rangle.$$

Substituting the results above gives us

$$\hat{\mathbf{a}} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

and

$$\hat{\mathbf{b}} = \langle \cos\phi, \sin\phi \rangle = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}.$$

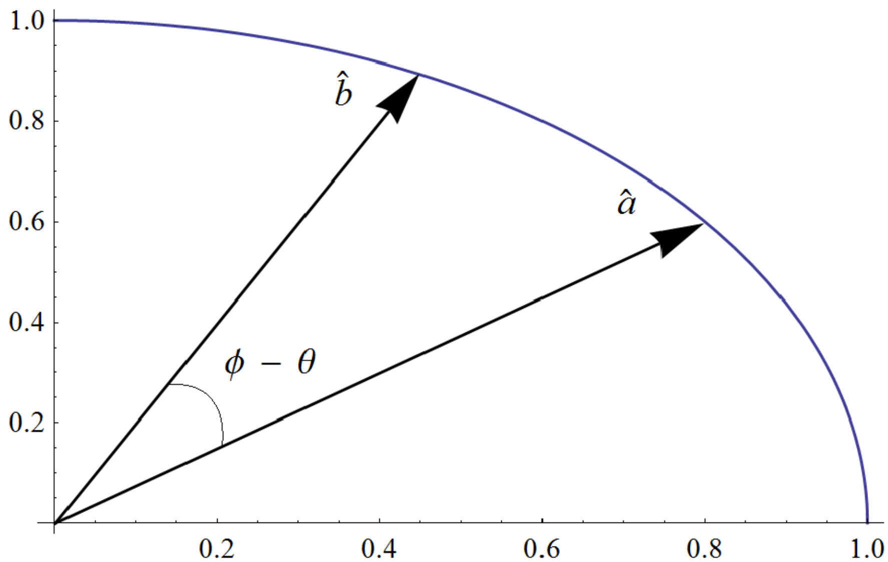

Figure 2: This figure shows the two unit vectors, $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, and the angle between them.

The dot product can be calculated in two ways: (1) multiply the magnitudes of the vectors together with the cosine of the angle between them and (2) multiply the respective components of the vectors and add them all up. Thus,

$$|\hat{\mathbf{a}}||\hat{\mathbf{b}}|\cos(\phi - \theta) = a_x b_x + a_y b_y$$

The magnitudes of the unit vectors are unity, so on the left side we're left with $\cos(\phi - \theta)$. On the right side we plug in the results calculated in the beginning.

$$\cos(\phi - \theta) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

Because cosine is an even function, that is

$$\cos(x) = \cos(-x),$$

we can write the left side as $\cos(\theta - \phi)$. Therefore,

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$