Problem 2.1

Time-dependent force

A 5-kg mass moves under the influence of a force $\mathbf{F} = (4t^2\hat{\mathbf{i}} - 3t\hat{\mathbf{j}})$ N, where t is the time in seconds (1 N = 1 newton). It starts at rest from the origin at $t = 0$. Find: (a) its velocity; (b) its position; and (c) $\mathbf{r} \times \mathbf{v}$, for any later time.

Solution

According to Newton's second law, if a force $\mathbf{F}$ acts on a mass m, it will have an acceleration $\mathbf{a}$.

$$\mathbf{F} = m\mathbf{a}$$

The acceleration vector is then

$$\begin{aligned}\mathbf{a}(t) &= \frac{1}{m}\mathbf{F} \\ &= \frac{1}{5}\langle 4t^2, -3t, 0\rangle \, \frac{\text{m}}{\text{s}^2} \\ &= \left\langle \frac{4}{5}t^2, -\frac{3}{5}t, 0 \right\rangle \frac{\text{m}}{\text{s}^2}.\end{aligned}$$

The velocity vector is obtained by integrating the acceleration vector with respect to time.

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\, dt \\ &= \left\langle \frac{4}{15}t^3 + C_1, -\frac{3}{10}t^2 + C_2, C_3 \right\rangle \frac{\text{m}}{\text{s}}\end{aligned}$$

Because the mass starts from rest, the initial condition for the velocity is $\mathbf{v}(0) = \langle 0, 0, 0\rangle$, so $C_1 = 0$, $C_2 = 0$, and $C_3 = 0$.

$$= \left\langle \frac{4}{15}t^3, -\frac{3}{10}t^2, 0 \right\rangle \frac{\text{m}}{\text{s}}$$

The position vector is obtained by integrating the velocity vector with respect to time.

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\, dt \\ &= \left\langle \frac{1}{15}t^4 + C_4, -\frac{1}{10}t^3 + C_5, C_6 \right\rangle \text{m}\end{aligned}$$

Because the mass starts at the origin, the initial condition for the position is $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, so $C_4 = 0$, $C_5 = 0$, and $C_6 = 0$.

$$= \left\langle \frac{1}{15}t^4, -\frac{1}{10}t^3, 0 \right\rangle \text{m}$$

Therefore,

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{1}{15}t^4 & -\frac{1}{10}t^3 & 0 \\ \frac{4}{15}t^3 & -\frac{3}{10}t^2 & 0 \end{vmatrix} = \left(-\frac{3}{150}t^6 + \frac{4}{150}t^6\right)\hat{\mathbf{z}} = \frac{t^6}{150}\hat{\mathbf{z}} \, \frac{\text{m}^2}{\text{s}}.$$