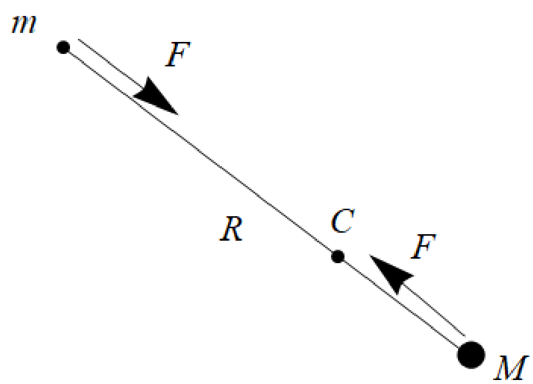
Problem 2.4

Circling particle and force

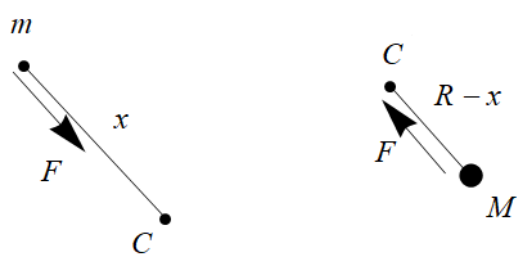
Two particles of mass m and M undergo uniform circular motion about each other at a separation R under the influence of an attractive constant force F. The angular velocity is ω radians per second. Show that $R = (F/\omega^2)(1/m + 1/M)$.

Solution

The situation is illustrated below.



Both masses rotate about the same point C along the line connecting them, but they are not equidistant from it unless the masses are equal. If mass m is at a distance x from C, then mass M is at a distance $R - x$ from C. Taking C to be the origin, draw the free-body diagrams for m and M.



Newton's second law states that force is equal to mass times acceleration.

$$\mathbf{F} = m\mathbf{a}$$

If we use polar coordinates, then this vector equation results in the following two scalar equations.

$$F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Applying these two equations for m and M, we get

Mass m	Mass M
$-F = m(-x\omega^2)$	$-F = M[-(R - x)\omega^2]$
$0 = m(0 + 0)$	$0 = M(0 + 0).$

Note that the negative sign on the left side of each equation is because the radial force points towards the origin. There are two equations for the two unknowns, x and R, so we can solve for both of them.

$$F = mx\omega^2$$

Divide both sides by $m\omega^2$.

$$x = \frac{F}{m\omega^2}$$

Substitute this result into the second equation.

$$-F = M\left[-\left(R - \frac{F}{m\omega^2}\right)\omega^2\right]$$

Divide both sides by $-M\omega^2$.

$$\frac{F}{M\omega^2} = R - \frac{F}{m\omega^2}$$

Solve for R.

$$R = \frac{F}{m\omega^2} + \frac{F}{M\omega^2}$$

Therefore,

$$R = \frac{F}{\omega^2}\left(\frac{1}{m} + \frac{1}{M}\right).$$

The quantity in parentheses is often written as $1/\mu$, where μ is known as the reduced mass.