

**Exercise 24**

Let  $\mathbf{b} = (3, 1, 1)$  and  $P$  be the plane through the origin given by  $x + y + 2z = 0$ .

- (a) Find an orthogonal basis for  $P$ . That is, find two nonzero orthogonal vectors  $\mathbf{v}_1, \mathbf{v}_2 \in P$ .
- (b) Find the orthogonal projection of  $\mathbf{b}$  onto  $P$ . That is, find  $\text{Proj}_{\mathbf{v}_1} \mathbf{b} + \text{Proj}_{\mathbf{v}_2} \mathbf{b}$ .

**Solution****Part (a)**

The equation of the plane can be written as

$$x + y + 2z = (1, 1, 2) \cdot (x, y, z) = 0.$$

The vector  $(1, 1, 2)$  is orthogonal to the plane at every point. Choose values for  $x$ ,  $y$ , and  $z$  that satisfy the equation, for example,  $x = 1$ ,  $y = 1$ , and  $z = -1$ . Let this be  $\mathbf{v}_1$ .

$$\mathbf{v}_1 = (1, 1, -1)$$

To get a second vector in the plane orthogonal to the first, take the cross product of  $(1, 1, 2)$  and  $\mathbf{v}_1$ .

$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -3\hat{\mathbf{x}} + 3\hat{\mathbf{y}} = (-3, 3, 0)$$

**Part (b)**

$$\begin{aligned} \text{Proj}_{\mathbf{v}_1} \mathbf{b} + \text{Proj}_{\mathbf{v}_2} \mathbf{b} &= (\mathbf{b} \cdot \hat{\mathbf{v}}_1)\hat{\mathbf{v}}_1 + (\mathbf{b} \cdot \hat{\mathbf{v}}_2)\hat{\mathbf{v}}_2 \\ &= \frac{\mathbf{b} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\mathbf{b} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 \\ &= \frac{(3, 1, 1) \cdot (1, 1, -1)}{1^2 + 1^2 + (-1)^2} (1, 1, -1) + \frac{(3, 1, 1) \cdot (-3, 3, 0)}{(-3)^2 + 3^2} (-3, 3, 0) \\ &= \frac{(3)(1) + (1)(1) + (1)(-1)}{3} (1, 1, -1) + \frac{(3)(-3) + (1)(3) + (1)(0)}{18} (-3, 3, 0) \\ &= \frac{3}{3} (1, 1, -1) + \frac{-6}{18} (-3, 3, 0) \\ &= (1, 1, -1) + (1, -1, 0) \\ &= (2, 0, -1) \end{aligned}$$